Chapter 6

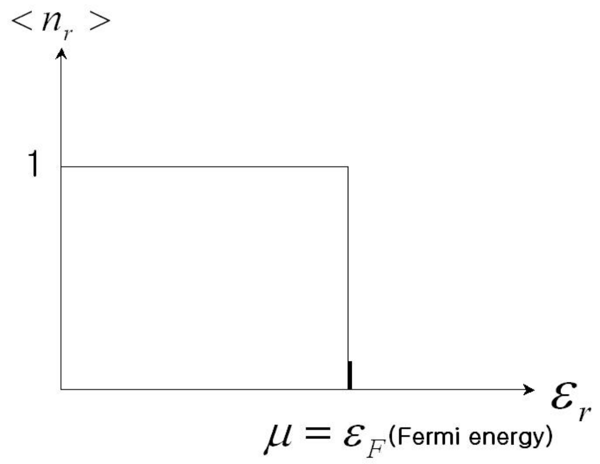
Quantum Fermi and Photon Gases

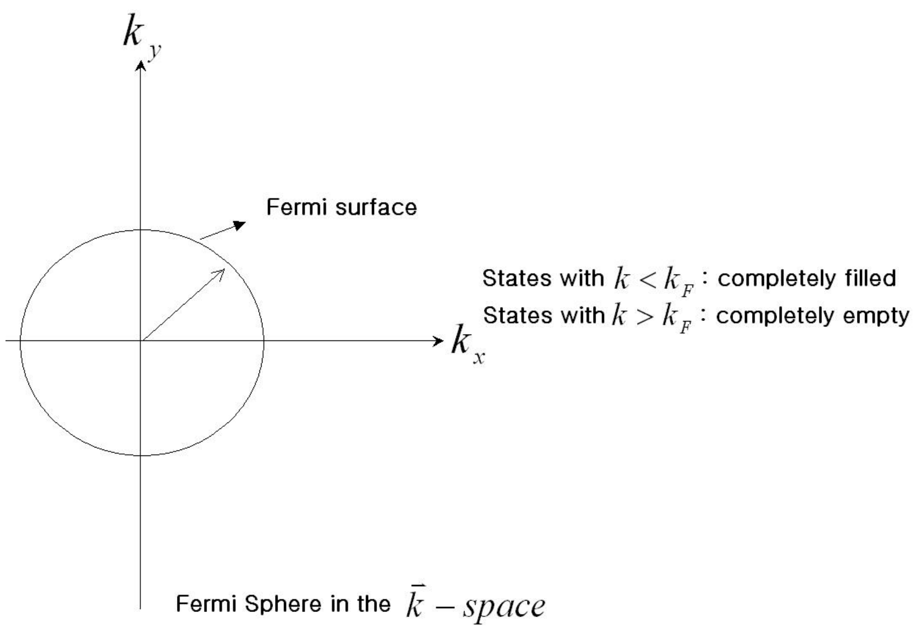
6.1 Free-Electron Gas

Firstly, let us consider the limiting case (T=0)

$$< n_r >= \frac{1}{e^{\beta(\varepsilon_r - \mu)} + 1} \quad (\beta \longrightarrow \infty),\ r = (\overrightarrow{k}, s),\ \varepsilon_r = \frac{\hbar^2 k^2}{2m} \tag{6.1}$$

$$\therefore \varepsilon_r < \mu \Longrightarrow < n_r >= 1, \quad \varepsilon_r > \mu \Longrightarrow < n_r >= 0 \tag{6.2}$$





Fermi Sphere in the $\vec{k} - space$

$$\text{Fermi Energy } \varepsilon_F \equiv \frac{\hbar^2 k_F^2}{2m} = \mu \quad \text{at } T = 0 \tag{6.3}$$

$$\sum_r < n_r >= N, \quad \varepsilon_r \propto V^{-2/3} \tag{6.4}$$

$$2 \cdot \frac{V}{(2\pi)^3} \cdot \left(\frac{4}{3}\pi k_F^3\right) = N \tag{6.5}$$

$$\therefore \varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} \cdot \left[3\pi^2 \cdot \frac{N}{V}\right]^{2/3} = \frac{\hbar^2}{2m} \cdot \left[3\pi^2 \cdot n\right]^{2/3} \tag{6.6}$$

$$P_r = -\frac{\partial \varepsilon_r}{\partial V} = \frac{2}{3} \cdot \frac{\varepsilon_r}{V} \tag{6.7}$$

$$P = \sum_r P_r \cdot < n_r >= \frac{2}{3} \cdot \frac{U}{V} \tag{6.8}$$

$$U =< E > \;=\; \sum_r \varepsilon_r < n_r > \tag{6.9}$$

$$= 2 \cdot \int_0^{k_F} \frac{d^3k}{\frac{(2\pi)^3}{V}} \cdot \frac{\hbar^2 k^2}{2m} \tag{6.10}$$

$$= 2 \cdot \frac{V}{(2\pi)^3} \cdot \frac{\hbar^2}{2m} \int_0^{k_F} dk \cdot 4\pi k^2 \cdot k^2 \tag{6.11}$$

$$= 2 \cdot \frac{V}{2\pi^2} \cdot \frac{\hbar^2}{2m} \int_0^{k_F} dk \cdot k^4 \tag{6.12}$$

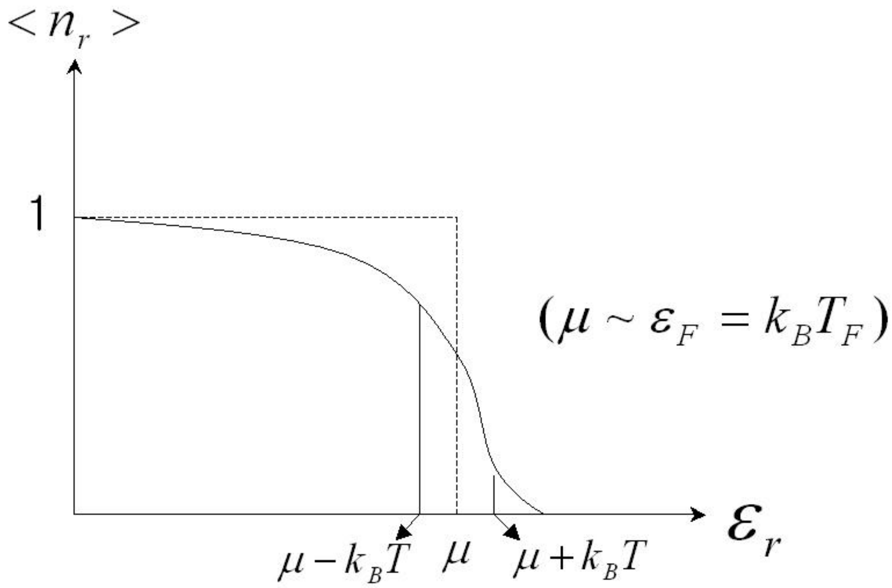
$$= 2 \cdot \frac{V}{2\pi^2} \cdot \frac{\hbar^2}{2m} \cdot \frac{k_F^5}{5} \tag{6.13}$$

$$= 2 \cdot \frac{V}{10\pi^2} \cdot k_F^3 \cdot \varepsilon_F = \frac{3N}{5} \cdot \varepsilon_F \tag{6.14}$$

$$\therefore u = U/N = \frac{3}{5}\varepsilon_F \tag{6.15}$$

$$\text{P(pressure)} = \frac{2}{3}U/V = \frac{2}{5}\varepsilon_F \cdot n, \quad n = N/V \tag{6.16}$$

Let's consider the case ($T \approx 0$)



$$\varepsilon_r \ll \mu \Longrightarrow < n_r > \simeq 1 \tag{6.17}$$

$$\varepsilon_r \gg \mu \Longrightarrow < n_r > \approx e^{-\beta(\varepsilon_r - \mu)} : \text{ classical Maxwell Boltzman dist.} \tag{6.18}$$

$$\varepsilon_r = \mu \longrightarrow < n_r > = 1/2 \tag{6.19}$$

$$\varepsilon_r = \mu + k_B T \longrightarrow < n_r > = 0.27 \tag{6.20}$$

$$\varepsilon_r = \mu - k_B T \longrightarrow < n_r > = 0.73 \tag{6.21}$$

Qualitative argument ($T \sim 0$)

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

No. of excited particles $\approx N \cdot \frac{T}{T_F}$

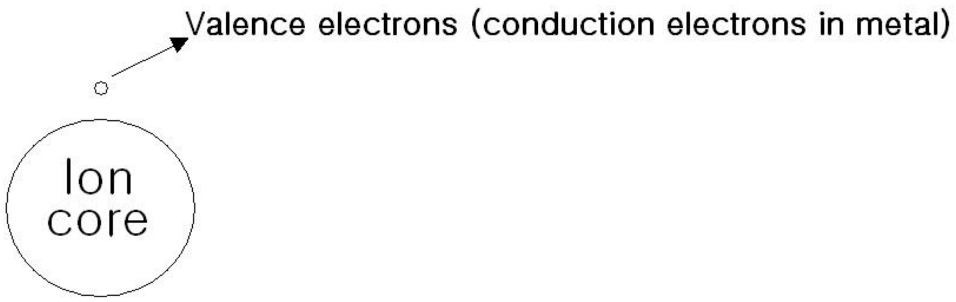
$$\therefore \Delta U \text{ (total excitation energy above the ground state)} \tag{6.22}$$

$$\wr\wr \tag{6.23}$$

$$(N \cdot \frac{T}{T_F}) \cdot (k_B T) \tag{6.24}$$

$$\therefore C_V \sim N \cdot k_B \cdot (\frac{T}{T_F}) \propto T \tag{6.25}$$

e.g. Conduction electrons in metals



$$\mathcal{H}_e = \sum_i \frac{\vec{p_i}^2}{2m} + \mathcal{H}_{int}(e-e) + \mathcal{H}_{int}(e-\text{core}) \quad (6.26)$$

$$\approx \sum_i \frac{\vec{p_i}^2}{2m} \longrightarrow \text{Conduction electrons in metal} \approx \text{free electron gas} \quad (6.27)$$

이 경우에, $s = \frac{\hbar}{2} \longrightarrow g(s) = 2$

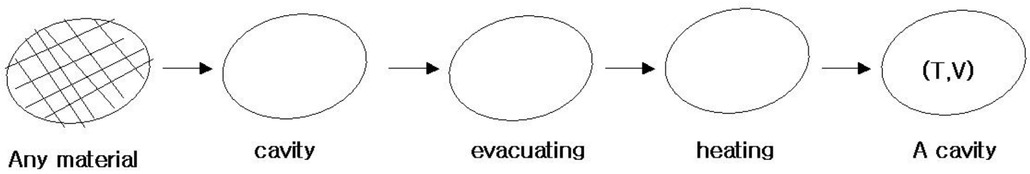
$$\therefore \varepsilon_F = \frac{\hbar^2}{2m}[3\pi^2 n]^{2/3} = k_B T_F, \quad n = N/V \quad (6.28)$$

typical T_F of metal?

$$n \sim 10^{23}/cm^3 : \ \text{metal} \longrightarrow T_F \sim 10^4 K - 10^5 K \sim 1 - 10 eV \quad (6.29)$$

$\therefore$ at room temperature (300K), $\frac{T}{T_F} \approx 10^{-2} \longrightarrow$ the electron gas is highly degenerate

6.2 Photon Statistics (Black Body Radiation)

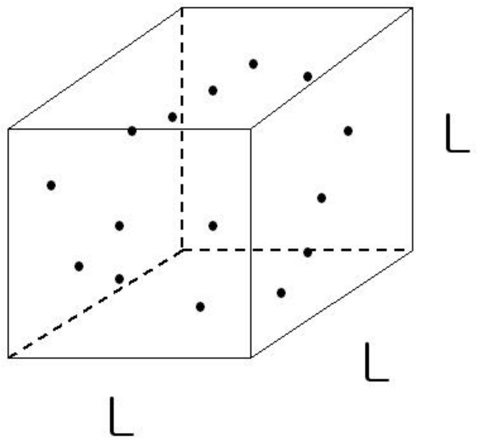


The atoms in the wall of this cavity will constantly emit and absorb electromagnetic radiation.

$$\downarrow$$

In equilibrium, there will be a certain amount of electromagnetic radiation in the cavity, and nothing else.

Note that if the cavity is sufficiently large, the thermodynamic properties of the radiation should be indep. of the nature of the wall. $\longrightarrow$ We can choose any bdry. condition that is convenient.



Periodic bdry. condi.

자유공간 (no matter) 에서의 Maxwell 방정식:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \cdot \frac{\partial \vec{B}}{\partial t}, \ \ \vec{\nabla} \times \vec{B} = \frac{1}{c} \cdot \frac{\partial \vec{E}}{\partial t} \tag{6.30}$$

$$\vec{\nabla} \cdot \vec{E} = 0, \ \ \vec{\nabla} \cdot \vec{B} = 0 \tag{6.31}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = 0 \tag{6.32}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \longrightarrow \nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{6.33}$$

$$\text{마찬가지로}, \nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0 \tag{6.34}$$

$$\vec{E} = \vec{E_0} \cdot e^{i(\vec{k} \cdot \vec{r} - \omega t)} \longrightarrow k^2 = \frac{\omega^2}{c^2} \longrightarrow \omega = ck \ \text{(dispersion relation)} \tag{6.35}$$

$$\vec{B} = \vec{B_0} \cdot e^{i(\vec{k} \cdot \vec{r} - \omega t)} \tag{6.36}$$

$$\vec{\nabla} \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right), \quad \vec{E_0} = (E_{0x}, E_{0y}, E_{0z}) \tag{6.37}$$

$$= i(E_{0z}k_y - E_{0y}k_z, E_{0x}k_z - E_{0z}k_x, E_{0y}k_x - E_{0y}k_y)e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{6.38}$$

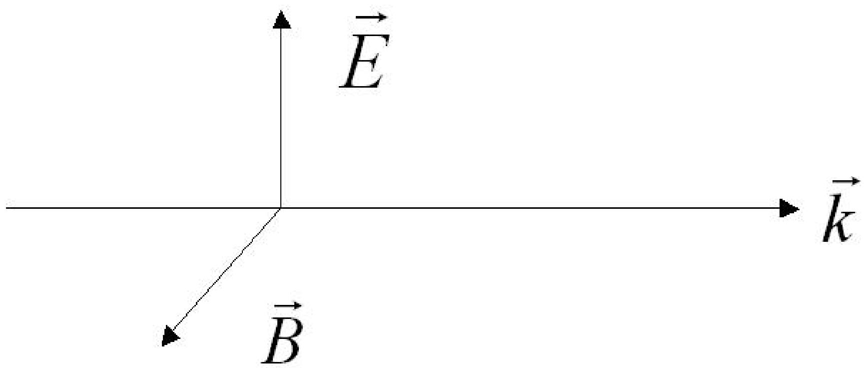
$$= i\vec{k} \times \vec{E} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \frac{i}{c}\omega \times \vec{B} = ik\vec{B} \tag{6.39}$$

$$\therefore \vec{B} = \hat{k} \times \vec{E} \tag{6.40}$$

$$\vec{\nabla} \cdot \vec{E} = i(E_{ox}k_x + E_{0y}k_y + E_{0z}k_z)e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0 \tag{6.41}$$

$$\therefore \vec{k} \cdot \vec{E} = 0 \tag{6.42}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \longrightarrow \vec{k} \cdot \vec{B} = 0 \tag{6.43}$$



(transverse light)

$\vec{E}$ (polarization vector) $\equiv \frac{\vec{E_0}}{E_0} \longrightarrow$ two indep. polarization (right circularly polarized, left circularly polarized)

$$r = (\vec{k}, \vec{\varepsilon}) \tag{6.44}$$

$$\downarrow$$

전자기파는 photon이라고 불리는 양자 (quantum)들로 이루어졌다. (Plank) photon: Energy$=\hbar\omega$

Momentum$=\hbar\overrightarrow{k}$, $k=\omega/c$

polarization $\overrightarrow{\varepsilon}$, $|\overrightarrow{\varepsilon}|$=1 & $\overrightarrow{k}\cdot\overrightarrow{\varepsilon}=0$

$\therefore$ 전자기파의 상태는 photon의 갯수로 나타낼 수 있다.

photon: indistinguishable, 똑같은 $(\overrightarrow{k},\overrightarrow{\varepsilon})$ 상태에 존재할 수 있는 photon의 수에 제약이 없다. $\longrightarrow$ photon: boson

Since atoms in the wall can emit and absorb photons, the number of photons in a cavity is not definite.

$$r=(\overrightarrow{k},\overrightarrow{\varepsilon}),\quad n_r:\ \text{No. of photons in the r-state.} \tag{6.45}$$

const. T & const. V $\longrightarrow F=F(T,V,N)$: min. in equilibrium.

In equilibrium, $N=\overline{N}$

Then, $\left(\frac{\partial F}{\partial N}\right)_{\overline{N}}=0\longrightarrow\mu=0\longrightarrow z=1$

$$\therefore <n_r>=\frac{1}{e^{\beta\varepsilon_r}-1},\quad \varepsilon_r=\hbar\omega,\quad r=(\overrightarrow{k},\overrightarrow{\varepsilon}) \tag{6.46}$$

$$U = \sum_r \hbar\omega <n_r> \tag{6.47}$$

$$= 2\cdot\sum_{\overrightarrow{k}}\hbar\omega\frac{1}{e^{\beta\hbar\omega}-1} \tag{6.48}$$

$$\sum_{\overrightarrow{k}}\longrightarrow\frac{V}{(2\pi)^3}\int d^3k=\frac{V}{(2\pi)^3}\int dk4\pi k^2=\frac{V}{2\pi^2c^3}\int d\omega\,\omega^2 \tag{6.49}$$

$$\therefore U = \frac{V\hbar}{\pi^2c^3}\int_0^\infty\frac{\omega^3}{e^{\beta\hbar\omega}-1}d\omega \tag{6.50}$$

$$\frac{U}{V} = \frac{\hbar}{\pi^2c^3}\int_0^\infty d\omega\frac{\omega^3}{e^{\beta\hbar\omega}-1} \tag{6.51}$$

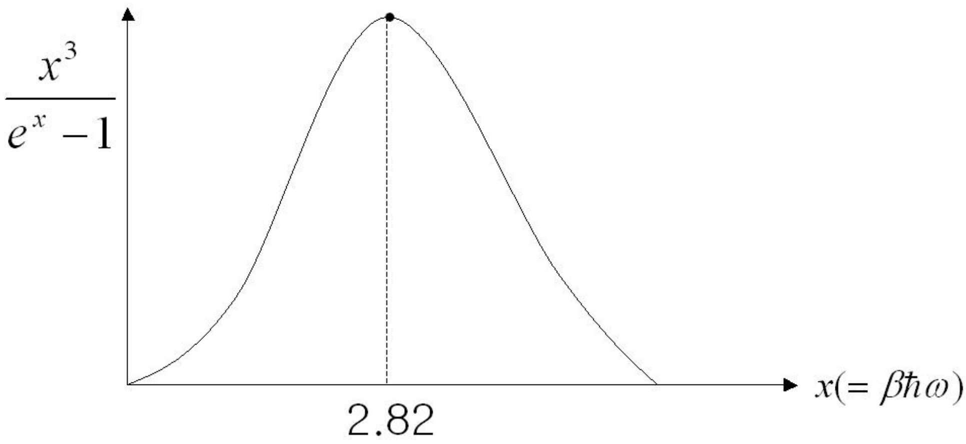
$$= \int_0^\infty d\omega u(\omega,T),\quad u(\omega,T)=\frac{\hbar}{\pi^2c^3}\frac{\omega^3}{e^{\beta\hbar\omega}-1} \tag{6.52}$$

$u(\omega, T) \cdot d\omega$: energy density due to photons whose frequencies lie bet. ω and

$\omega + d\omega \longrightarrow$ radiation formula of Plank

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3} \frac{1}{(\beta\hbar)^4} \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{(k_B T)^4}{\pi^2 c^3 \hbar^3} \cdot \int_0^\infty dx \frac{x^3}{e^x - 1}, \quad x = \beta\hbar\omega \quad (6.53)$$

$$u(\omega, T) d\omega = u(x, T) dx, \;\; u(x, T) = \frac{(k_B T)^4}{\pi^2 c^3 \hbar^3} \frac{x^3}{e^x - 1} \quad (6.54)$$



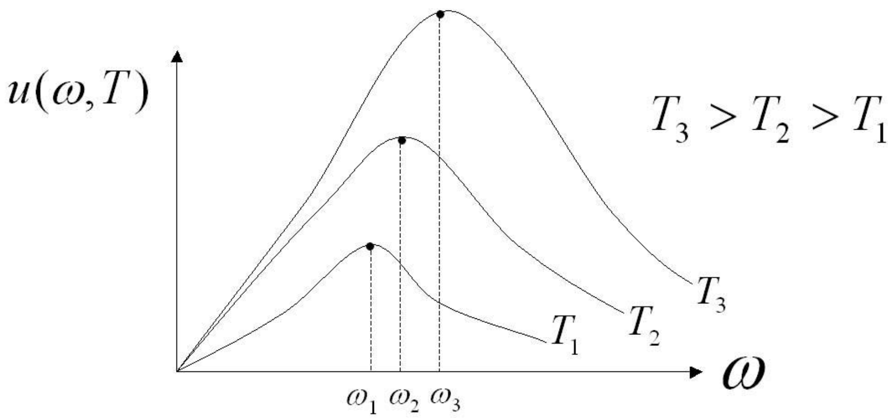
$$\left(\frac{x^3}{e^x - 1}\right)' = \frac{3x^2}{e^x - 1} - \frac{x^3 e^x}{(e^x - 1)^2} = \frac{3x^2(e^x - 1) - x^3 e^x}{(e^x - 1)^2} = 0 \qquad (6.55)$$

$$\therefore 3(e^x - 1) - x e^x = 0 \qquad (6.56)$$

$$(3 - x) \cdot e^x = 3 \longrightarrow x \approx 2.82 \qquad (6.57)$$

즉, $x = \beta\hbar\omega \approx 2.82$에서 max $\longrightarrow \omega_{max} \cong 2.82 \times \frac{k_B T}{\hbar}$

$\therefore$ as T increases, ω_{max} increases



$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \int_0^\infty dx \cdot x^3 \frac{e^{-x}}{1-e^{-x}} = \int_0^\infty dx \cdot x^3 e^{-x} (\sum_{l=0}^\infty e^{-lx}) \quad (6.58)$$

$$= \sum_{l=1}^\infty \int_0^\infty dx \cdot x^3 e^{-lx} \quad (6.59)$$

$$= \sum_{l=1}^\infty \frac{1}{l^4} \int_0^\infty dy \cdot y^3 e^{-y}, \quad (y = lx) \quad (6.60)$$

$$= 6 \cdot \zeta(4), \quad \zeta(4) = \frac{\pi^4}{90} \quad (6.61)$$

$$= \frac{\pi^4}{15} \quad (6.62)$$

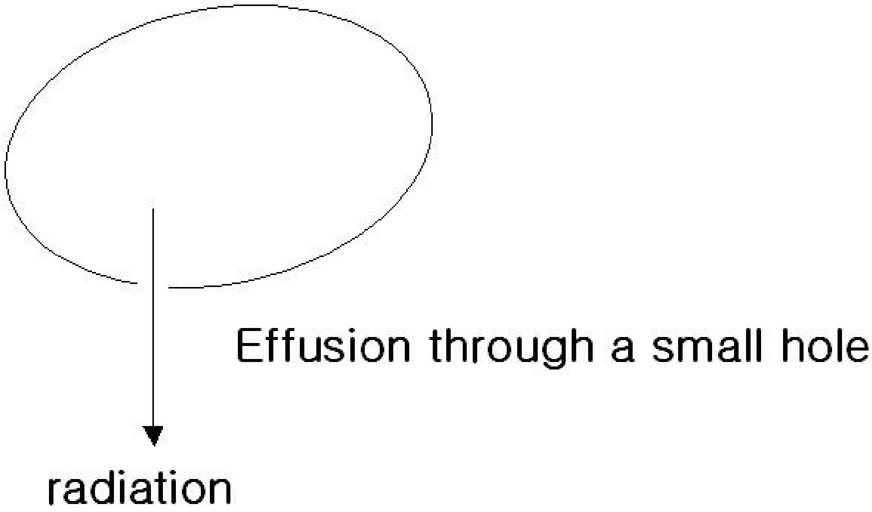
$$\therefore \frac{U}{V} = \frac{(k_B T)^4}{\pi^2 \hbar^3 c^3} \cdot \frac{\pi^4}{15} = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} \cdot T^4 \quad (6.63)$$

$$C_V : \text{ specific heat per unit volume} \quad (6.64)$$

$$C_V = \left(\frac{\partial (U/V)}{\partial T} \right)_V = \frac{4\pi^2 k_B^4}{15(\hbar c)^3} \cdot T^3 \quad (6.65)$$

Note that C_V is not bounded as $T \longrightarrow \infty$

How to check these facts experimentally?



$I(T)$: the amount of total energy per second per unit area

Φ (No. of particles per unit area per second)$=\frac{1}{4}\cdot n\cdot\overline{V}$

$\overline{V}$: 입자의 평균속도

n: 입자밀도

$$\therefore I(T)=\frac{1}{4}c\cdot\frac{U}{V}=\frac{\pi^2k_B^4}{60\hbar^3c^2}T^4=\sigma\cdot T^4\ :\ \text{Stephan's Law} \tag{6.66}$$

$$\sigma=\frac{\pi^2k_B^4}{60\hbar^3c^2}\ :\ \text{Stephan's const.} \tag{6.67}$$

P (radiation pressure)

$$U = \sum_r \varepsilon_r < n_r >,\quad \varepsilon_r=\hbar\omega=\hbar ck,\quad r=(\overrightarrow{k},\overrightarrow{\varepsilon}) \tag{6.68}$$

$$\overrightarrow{k} = \frac{2\pi}{L}\overrightarrow{n},\quad k^2=\left(\frac{2\pi}{L}\right)^2\cdot n^2\longrightarrow k=c\cdot L^{-1}=\frac{c}{V^{1/3}}\ (c:\ \text{const.}) \tag{6.69}$$

$$\therefore \varepsilon_r=\text{const.}\cdot V^{-1/3},\quad P_r=-\frac{\partial\varepsilon_r}{\partial V}=\frac{1}{3}\frac{\varepsilon_r}{V} \tag{6.70}$$

$$\therefore P=\sum_r P_r\cdot < n_r > = \frac{1}{3}\cdot\frac{U}{V}=\frac{\pi^2k_B^4}{45(\hbar c)^3}\cdot T^4\ \text{indep. of V} \tag{6.71}$$

$\overline{N}$?

$$\overline{N} = \sum_r < n_r >= 2 \cdot \frac{V}{2\pi^2 c^3} \int_0^\infty d\omega \cdot \omega^2 \cdot \frac{1}{e^{\beta\hbar\omega} - 1}, \quad (x = \beta\hbar\omega) \tag{6.72}$$

$$\overline{N} = \frac{V}{\pi^2 c^3} \cdot \frac{1}{(\beta\hbar)^3} \int_0^\infty dx \cdot \frac{x^2}{e^x - 1} \tag{6.73}$$

$$= \frac{V(k_B T)^3}{\pi^2 (\hbar c)^3} \int_0^\infty dx \cdot \frac{x^2}{e^x - 1} \tag{6.74}$$

$$\int_0^\infty dx \cdot \frac{x^2}{e^x - 1} = \int_0^\infty dx \cdot x^2 e^{-x} (1 - e^{-x})^{-1} = \sum_{l=1}^\infty \int_0^\infty dx \cdot x^2 e^{-lx}, \quad (y = lx) \tag{6.75}$$

$$= \sum_{l=1}^\infty l^{-3} \int_0^\infty dy \cdot y^2 e^{-y} \tag{6.76}$$

$$= 2 \cdot \zeta(3) \tag{6.77}$$

$$\overline{N} = V \cdot \frac{2k_B^3 \zeta(3)}{\pi^2 (\hbar c)^3} \cdot T^3 \tag{6.78}$$

Can $\overline{N}$ be a representative value about N?

$$(\Delta N)^2 = \overline{N} k_B T \frac{\overline{N}}{V} \kappa_T \tag{6.79}$$

$$\kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T \tag{6.80}$$

$$\frac{\partial P}{\partial V} = 0 \longrightarrow (\kappa_T \longrightarrow \infty) \tag{6.81}$$

$\therefore$ fluctuation can not be neglected

$\therefore$ No. of photon is indefinite

C. E.

$$Q_{B.E.} = \sum_{\{n_r\}, \sum n_r = N} e^{-\beta \sum_r n_r \cdot \varepsilon_r}, \quad n_r = 0, 1, 2, \cdots \tag{6.82}$$

$\downarrow$

$$Q_{ph.} = \sum_{\{n_r\}} e^{-\beta \sum_r n_r \cdot \varepsilon_r} = (\sum_{n_1} e^{-\beta n_1 \varepsilon_1})(\sum_{n_2} e^{-\beta n_2 \varepsilon_2}) \cdots \tag{6.83}$$

$$= \prod_r \frac{1}{1 - e^{-\beta \varepsilon_r}} \tag{6.84}$$

$$\therefore \ln Q_{ph.} = -\sum_r \ln(1 - e^{-\beta \varepsilon_r}) \tag{6.85}$$

$$< n_r > = -\frac{1}{\beta} \frac{\partial \ln Q_{ph.}}{\partial \varepsilon_r} \tag{6.86}$$

$$= \frac{1}{e^{\beta \varepsilon_r} - 1} \tag{6.87}$$

$$\text{cf. B. E.} \longrightarrow < n_r >= \frac{1}{e^{\beta(\varepsilon_r - \mu)} - 1} \tag{6.88}$$

$$\therefore \mu = 0 \longrightarrow z = 1 \text{ for photon} \tag{6.89}$$