Chapter 8

Perturbation Theory

8.1 섭동론

$$H = H_0 + \varepsilon H_1, \quad \varepsilon \ll 1 \tag{8.1}$$

H_0: 비섭동 Hamiltonian, 풀이를 안다. εH_1: 섭동항

행성운동

H_0: 행성과 태양만을 고려한 역학계

예를 들어 금성의 운동

목성의 인력 평균 크기: 태양 인력의 2×10^{-5}

지구의 인력 평균 크기: 태양 인력의 4×10^{-6}

$\rightarrow$ 섭동으로 취급

다변수 비선형계 $\rightarrow$ 복잡한 계

1차원 1개 자유도의 Hamiltonian 계

보기. 간단한 1차원 1 변수 역학계

$$\frac{dx}{dt} = x + \varepsilon x^2, \quad \text{초기 조건} x(0) = A \tag{8.2}$$

$$x(t) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \varepsilon^2 x^{(2)}(t) + \cdots \tag{8.3}$$

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon + \frac{f''(x)}{2!}\varepsilon^2 + \cdots \tag{8.4}$$

ε이 크면 diverge. 모든 ε에 대해서도 diverge하는 경우가 있다.

$$\dot{x}^{(0)} + \varepsilon \dot{x}^{(1)} + \varepsilon^2 \dot{x}^{(2)} = x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \varepsilon ({x^{(0)}}^2 + 2\varepsilon x^{(0)} x^{(1)}) + O(\varepsilon^3) \tag{8.5}$$

모든 ε에 대해서 성립하려면

$$\begin{aligned}\dot{x}^{(0)} &= x^{(0)} &(8.6)\\ \dot{x}^{(1)} &= x^{(1)} + {x^{(0)}}^2 &(8.7)\\ \dot{x}^{(2)} &= x^{(2)} + 2x^{(0)}x^{(1)} &(8.8)\end{aligned}$$

Eq. (8.3) $\rightarrow x(0) = A \rightarrow x^{(0)}(0) = A, A^{(k)} = 0, k \geq 1$

$$\begin{aligned}x^{(0)} &= Ae^t &(8.9)\\ \dot{x}^{(1)} &= x^{(1)} + A^2e^{2t},\ \ x^{(1)}(0) = 0 &(8.10)\\ x^{(1)} &= A^2e^t(e^t-1) \rightarrow x^{(2)} = A^3e^t(e^t-1)^2 &(8.11)\\ \frac{dy}{dx} &+ P(x)y = Q(y) &(8.12)\\ y(x) &= e^{-\int P(x)dx}\int Q(x)e^{\int p(x)dx}dx + ce^{-\int p(x)dx}(c : \text{적분상수}) &(8.13)\end{aligned}$$

• 섭동론: <u>모든 낮은 차수의 섭동 보정을 알면 점차적으로 계산할 수 있다.</u>

2차까지의 섭동보정은

$$x(t) = Ae^t[1 + \varepsilon A(e^t-1) + \varepsilon^2A^2(e^t-1)^2] + O(\varepsilon^3) \tag{8.14}$$

$$\varepsilon A(e^t-1) \quad t \rightarrow \infty \tag{8.15}$$

같은 정밀도의 섭동보정을 요철할 때 시간이 길면 길수록 많은 보정항을 넣어야 한다.

Exact solution:

$$\begin{aligned}\int \frac{dx}{x+\varepsilon x^2} &= \int dt &(8.16)\\ \ln\frac{\varepsilon x}{\varepsilon x+1} &= t + c' &(8.17)\\ \frac{\varepsilon x}{\varepsilon x+1} &= ce^t, \quad t = 0 \rightarrow x = A &(8.18)\\ x(t) &= \frac{Ae^t}{1-\varepsilon A(e^t-1)} &(8.19)\end{aligned}$$

이 멱급수의 수렴반경은

$$\underline{\varepsilon A(e^t - 1) = 1}, \quad t_c = \ln \frac{1 + A\varepsilon}{A\varepsilon} \tag{8.20}$$

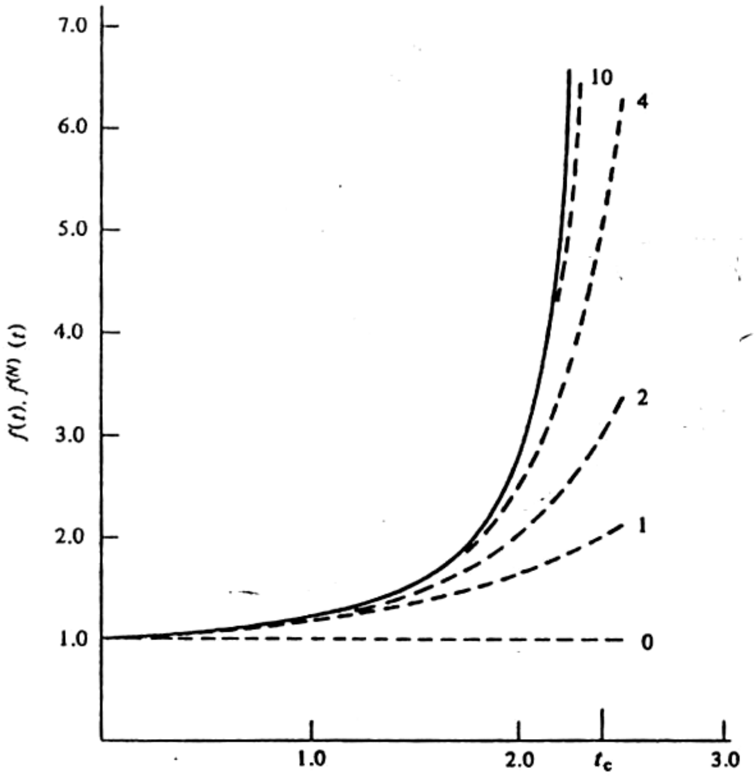
$t < t_c$에만 수렴하고, $t > t_c$에서 이 멱급수는 아무 의미가 없다.

Exact quotient:

$$f(t) = \frac{x(t)}{x^{(0)}(t)} = \frac{x(t)}{Ae^t} = \frac{1}{1 - \varepsilon A(e^t - 1)} \tag{8.21}$$

N차 섭동보정에 따른 근사몫(approximation quotient)

$$f^{(N)}(t) = \sum_{n=0}^{N} [\varepsilon A(e^t - 1)]^n \tag{8.22}$$



$$\dot{x} = x + \varepsilon x^2 \to H \tag{8.23}$$

$$\dot{x} = x \to H_0 \tag{8.24}$$

ε이 아무리 작아도 유한하면 섭동계에서는 운동이 자연경계 $t = t_c$에서 $x = \infty$

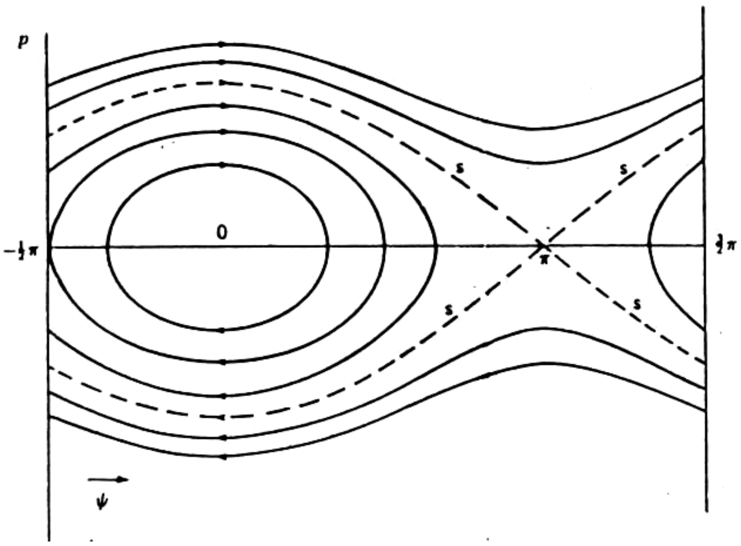
비섭동계에는 자연경계가 없다.

t가 작을 때는 이러한 난점에도 불구하고 유용하게 쓰인다.

8.2 Hamiltonian System

$$H(q,p) = H_0(q,p) + \varepsilon H_1(q,p) \tag{8.25}$$

• $H_0(q,p)$계의 풀이를 바탕으로 한 섭동전개를 하므로 비섭동과 섭동운동은 같은 종류이어야 한다. •



면적흔들이를 보기들면,

$$H(p,\psi) = \frac{p^2}{2} - \alpha^2 \cos\psi \tag{8.26}$$

빠른 회전을 비섭동으로 보면

$$\frac{p^2}{2} \gg \alpha^2 \tag{8.27}$$

$$H_0 = \frac{p^2}{2}, \quad H_1 = -\alpha^2 \cos^2\psi \tag{8.28}$$

$$\text{나중에 } \varepsilon = 1.$$

원점 근방에서 작은 진동을 보면, $\psi \ll 1$

$$H(p,\psi) = \frac{1}{2}(p^2 + \alpha^2\psi^2) - \alpha^2 - \alpha^2(\cos\psi - 1 + \frac{\psi^2}{2}) \tag{8.29}$$

$$\downarrow \ \text{drop the constant term} - \alpha^2$$

$$H = \bar{H}(p,\psi) + \alpha^2 = \bar{H}_0 + \varepsilon \bar{H}_1 \tag{8.30}$$

$$\bar{H}_0 = \frac{1}{2}(p^2 + \alpha^2\psi^2) \tag{8.31}$$

$$\bar{H}_1 = -\alpha^2(\cos\psi - 1 + \frac{\psi^2}{2}) : O(\psi^4) \tag{8.32}$$

일반론

1st order approximation

$$H = H_0(J) + \varepsilon H_1(J,\phi) \tag{8.33}$$

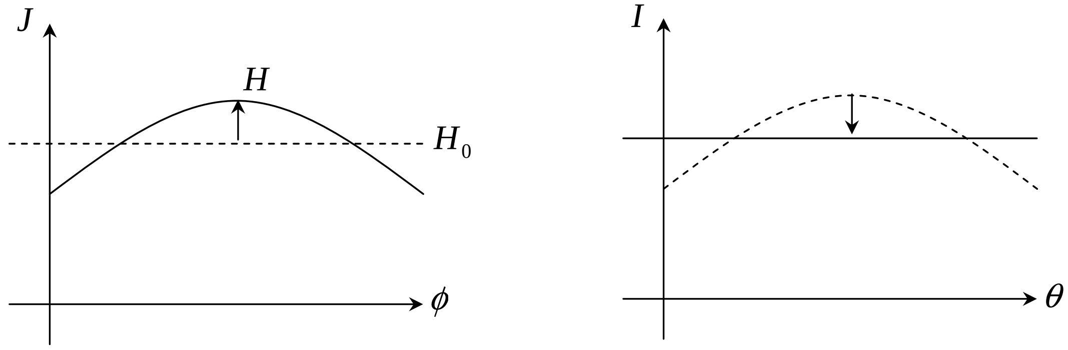
H_0의 각-작용량 변수 ϕ, J로 쓰자.

H의 각-작용량 변수 θ, I로 쓰자.

(J,ϕ): old action-angle variables

$\downarrow$

(I,θ): new action-angle variables



$$\phi = \phi(\theta, I) = \phi^{(0)}(\theta, I) + \varepsilon\phi^{(1)}(\theta, I) + O(\varepsilon^2) \tag{8.34}$$

$$J = J(\theta, I) = J^{(0)}(\theta, I) + \varepsilon J^{(1)}(\theta, I) + O(\varepsilon^2) \tag{8.35}$$

$$H_0(q,p) \quad \to \quad K_0(J) \tag{8.36}$$

$$(\phi, J)$$

$$q = q(\phi, J) \quad p = p(\phi, J) \tag{8.37}$$

$$H_1(q,p) \to H_1(\phi, J) \tag{8.38}$$

$$H(\phi, J) = H_0(J) + \varepsilon H_1(\phi, J) \tag{8.39}$$

$$H(q,p) \quad \to \quad H(I) \tag{8.40}$$

$$(\theta, I)$$

As $\varepsilon \to 0$, $\phi \to \theta$ and $J \to I$

$$\phi^{(0)}(\theta, I) = \theta, \quad J^{(0)}(\theta, I) = I \tag{8.41}$$

New Hamiltonian:

$$K = K_0(I) + \varepsilon K_1(I) \tag{8.42}$$

$(\phi, J) \to (\theta, I)$는 정준 변환

(1) $\dfrac{\partial(\phi, J)}{\partial(\theta, I)} = 1$

(2) $J^{(1)}$도 θ의 주기 함수 period 2π

$\phi^{(1)}$도 θ의 주기 함수

$$2\pi I = \int_0^{2\pi} I d\theta = \int J d\phi \tag{8.43}$$

$$\begin{aligned} H(\phi, J) &= H_0(H) + \varepsilon H_1(\phi, J) + O(\varepsilon^2) & (8.44) \\ K(I) &= H_0(J^{(0)} + \varepsilon J^{(1)}) + \varepsilon H_1(\theta, I) + O(\varepsilon^2) & (8.45) \\ &= H_0(I) + \frac{\partial H_0}{\partial I}\varepsilon J^{(1)} + \varepsilon H_1(\theta, I) + O(\varepsilon^2) & (8.46) \\ &= K_0 + \varepsilon K_1 + O(\varepsilon^2) & (8.47) \end{aligned}$$

$$\begin{aligned} K_0 &= H_0(I) & (8.48) \\ K_1 &= J^{(1)}\frac{\partial H_0}{\partial I} + H_1(\theta, I) & (8.49) \end{aligned}$$

$$\phi = \theta + \varepsilon\phi^{(1)}(\theta, I) \to d\phi = d\theta(1 + \varepsilon\frac{\partial \phi^{(1)}}{\partial \theta}) \quad (8.50)$$

$\downarrow$

$$I = \frac{1}{2\pi}\int_0^{2\pi}(J^{(0)} + \varepsilon J^{(1)})d\phi \quad (8.51)$$

$$\int_0^{2\pi} J^{(1)}d\theta = \int_0^{2\pi}\frac{\partial \phi^{(1)}}{\partial \theta}d\theta = 0 \quad (8.52)$$

$$\frac{1}{2\pi}\int K_1(I)d\theta = K_1(I) \quad (8.53)$$

$$\text{r.h.s.} = \frac{1}{2\pi}\frac{\partial H_0}{\partial I}\int J^{(1)}d\theta + \frac{1}{2\pi}\int_0^{2\pi} H_1(\theta, I)d\theta \quad (8.54)$$

$$\therefore K_1(I) = \frac{1}{2\pi}\int_0^{2\pi} H_1(\theta, I)d\theta : \text{mean perturbation} \quad (8.55)$$

$$J^{(1)} = \frac{K_1(I) - H_1(\theta, I)}{\omega_0(I)}, \quad \omega_0(I) = \frac{\partial H_0}{\partial I} \quad (8.56)$$

$\phi^{(1)}(\theta, I)$의 상수 불확실성을

$$\frac{1}{2\pi}\int \phi^{(1)}(\theta, I)d\theta = 0 \quad (8.57)$$

이 되게 정한다.

$$\begin{vmatrix} \frac{\partial \phi}{\partial \theta} & \frac{\partial \phi}{\partial I} \\ \frac{\partial J}{\partial \theta} & \frac{\partial J}{\partial I} \end{vmatrix} = 1 \tag{8.58}$$

$$\begin{vmatrix} \frac{\partial \phi^{(1)}}{\partial \theta} + \varepsilon \frac{\partial \phi^{(1)}}{\partial \theta} & \varepsilon \frac{\partial \phi^{(1)}}{\partial I} \\ \varepsilon \frac{\partial J^{(1)}}{\partial \theta} & 1 + \varepsilon \frac{\partial J^{(1)}}{\partial I} \end{vmatrix} = 1 \tag{8.59}$$

$$\therefore \frac{\partial \phi^{(1)}}{\partial \theta} = -\frac{\partial J^{(1)}}{\partial I} \tag{8.60}$$

$$\phi^{(1)} = -\int \frac{\partial J^{(1)}}{\partial I} d\theta + const. \tag{8.61}$$

c는 $\int_0^{2\pi} \phi^{(1)} d\theta = 0$이 되게 정한다.

- Fast rotation

$$H_0 = \frac{p^2}{2}, \quad H_1 = -\alpha^2 \cos\psi \tag{8.62}$$

$$J = p, \ \ \phi = \psi \rightarrow H_1 = -\alpha^2 \cos\phi \tag{8.63}$$

$$K_1(I) = \frac{1}{2\pi} \int_0^{2\pi} H_1(I, \theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (-\alpha^2) \cos\theta d\theta = 0 \tag{8.64}$$

$$\underline{K_1(I) = 0} \tag{8.65}$$

perturbation에서 주기에 대해서 평균했을 때 0이 1차 섭동에너지

$$H_0 = \frac{I^2}{2}, \quad \omega_0 = \frac{\partial H_0}{\partial I} = I \tag{8.66}$$

$$J^{(1)} = +\alpha^2 \frac{\cos\theta}{I} \tag{8.67}$$

$$\frac{\partial J^{(1)}}{\partial I} = \alpha^2 \frac{\cos\theta}{I} \tag{8.68}$$

$$\phi^{(1)} = -\int \frac{\partial J^{(1)}}{\partial I} d\theta = \frac{\alpha^2 \sin^2\theta}{I^2} \tag{8.69}$$

$$K(I) = \frac{I^2}{2} + O(\alpha^4), \tag{8.70}$$

$$\phi(\theta, I) = \theta + \frac{\alpha^2 \sin^2\theta}{I^2} + O(\alpha^4) \tag{8.71}$$

$$J(\theta, I) = I + \frac{\alpha^2 \cos\theta}{I^2} \tag{8.72}$$

$$\Omega = \omega_0 = I, \quad \theta = It + \delta \tag{8.73}$$

$$p = J = I + \frac{\alpha^2 \cos(It + \delta)}{I} + O(\alpha^4) \tag{8.74}$$

$$\psi = \phi = It + \delta + \frac{\alpha^2 \sin(It + \delta)}{I^2} + O(\alpha^4) \tag{8.75}$$

$$E = K(I) = \frac{I^2}{2} + O(\alpha^4) \tag{8.76}$$

- 미소진동근방에서의 섭동

$$H_0 = \frac{1}{2}(p^2 + \alpha^2\psi^2) \tag{8.77}$$

$$H_1 = -\alpha^2(\cos\psi - 1 + \frac{1}{2}\psi^2) \tag{8.78}$$

$$\simeq -\alpha^2\frac{\psi^4}{24} + O(\psi^6) \tag{8.79}$$

$$H_0(p, \psi) \quad \rightarrow \quad H_0(J) = \alpha J \tag{8.80}$$

$$(\psi, J)$$

$$\psi = \left(\frac{2J}{\alpha}\right)^{1/2} \sin\psi \tag{8.81}$$

$$p = (2J\alpha)^{1/2} \cos\psi \tag{8.82}$$

$$H_1(\psi, J) = -\frac{1}{6}J^2 \sin^4\psi \tag{8.83}$$

$$K_0(I) = \alpha I, \quad K_1(I) = -\frac{I^2}{12\pi}\int_0^{2\pi} \sin^4\theta d\theta = -\frac{I^2}{16} \tag{8.84}$$

$$\int_0^{2\pi} \sin^4\theta d\theta = -\frac{I^2}{12\pi}\int_0^{2\pi} \frac{3 - 4\cos 2\theta + \cos 4\theta}{4} d\theta \tag{8.85}$$

$$\sin^4\theta = \left(\frac{1 - \cos 2\theta}{2}\right)^2 \tag{8.86}$$

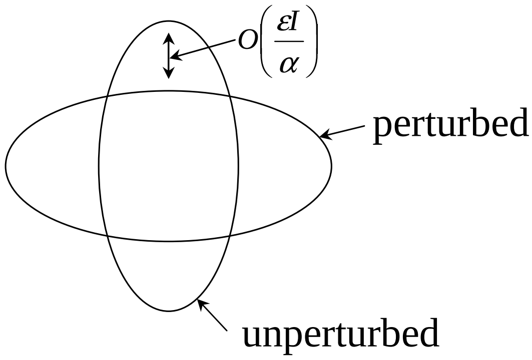
$$\underline{K(I) = \alpha I - \varepsilon\frac{I^2}{16} + O(\varepsilon^2)} \tag{8.87}$$

$$
\begin{aligned}
J^{(1)} &= \frac{-\frac{I^2}{16} + \frac{I^2}{6}\sin^4\theta}{\alpha} = -\frac{I^2}{\alpha}(\frac{1}{16} - \frac{\sin^4\theta}{6}) && (8.88)\\
&= -\frac{I^2}{\alpha}(\frac{1}{16} - \frac{3 - 4\cos 2\theta + \cos 4\theta}{48}) && (8.89)\\
&= -\frac{I^2}{12\alpha}[\cos 2\theta - \frac{1}{4}\cos 4\theta] && (8.90)\\
-\frac{\partial J^{(1)}}{\partial I} &= \frac{I}{\alpha}(\frac{1}{8} - \frac{\sin^4\theta}{3}) = \frac{I}{6\alpha}[\cos 2\theta - \frac{1}{4}\cos 4\theta] && (8.91)\\
\phi^{(1)} &= \frac{I}{\alpha}\frac{4\frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta}{2\phi} = -\int\frac{\partial J^{(1)}}{\partial I}d\theta && (8.92)\\
&= \frac{I}{12\alpha}(\sin 2\theta - \frac{\sin 4\theta}{8}) && (8.93)\\
\frac{\partial K(I)}{\partial I} &= \Omega = \alpha - \varepsilon\frac{I}{8} + O(\varepsilon^2) && (8.94)\\
\psi &= (\frac{2I}{\alpha})^{1/2}(1 - \frac{\varepsilon I}{12}(\cos 2\theta - \frac{\cos 4\theta}{4})^{1/2} && (8.95)\\
&\quad \times\sin\{\theta + \frac{\varepsilon I}{12\alpha}(\sin 2\theta - \frac{\sin 4\theta}{8}\}) && (8.96)\\
p &= (2\alpha I)^{1/2}\{1 - \frac{\varepsilon I}{12}(\cos 2\theta - \frac{\cos 4\theta}{4}\}^{1/2} && (8.97)\\
&\quad \times\cos\{\theta + \frac{\varepsilon I}{12\alpha}(\sin 2\theta - \frac{\sin 4\theta}{8})\} && (8.98)\\
&\quad \theta = \Omega t + \delta && (8.99)
\end{aligned}
$$

$$
\begin{aligned}
\psi &= (\frac{2I}{\alpha})^{1/2}[\sin\theta + \frac{\varepsilon I}{48\alpha}(3\sin\theta + \frac{1}{2}\sin 3\theta)] && (8.100)\\
p &= (2\alpha I)^{1/2}[\cos\theta - \frac{\varepsilon I}{32\alpha}(2\cos\theta - 3\cos 3\theta)] && (8.101)
\end{aligned}
$$

T 시간 동안

$$\phi = \alpha T \tag{8.102}$$



perturbed

$$\theta = \Omega T = (\alpha - \varepsilon \frac{I}{6})T \quad (8.103)$$

$$T = \frac{8\pi}{\varepsilon I} \quad (8.104)$$

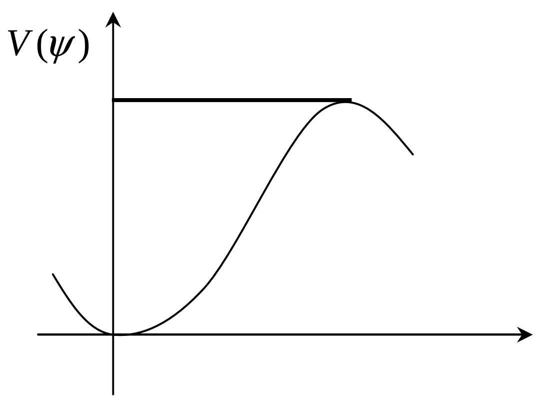
$$-(\theta - \phi) = \pi \quad (8.105)$$

$\varepsilon = 1$인 경우 $\rightarrow I = 8\alpha \rightarrow \Omega = 0$ (separatrix) $\rightarrow$ small denominator ($J^{(1)}$ 식을 보시오)

<u>Separatrix energy</u>

$$K(8\alpha) = \alpha(8\alpha) - \frac{64}{16}\alpha^2 = 4\alpha^2(\text{poor result}) \quad (8.106)$$

$$\begin{aligned} H &= \frac{1}{2}(p^2 + \alpha^2\psi^2) - \alpha^2\psi^4/24 \quad (8.107) \\ &= \frac{1}{2}p^2 + \alpha^2(\frac{\psi^2}{2} - \psi^4) \quad (8.108) \end{aligned}$$



$$E_s = \alpha^2(\frac{6}{2} - \frac{36}{24}) = \frac{3}{2}\alpha^2 \tag{8.109}$$

$$H = \frac{p^2}{2} - \alpha^2 \cos\psi \rightarrow E_s = \alpha^2 \tag{8.110}$$

- 제2차 섭동론

면적보존:

$$\frac{\partial(\Phi, J)}{\partial(\theta, I)} = 1 \tag{8.111}$$

Φ, J: θ의 periodic function

$$H_0 = H_0(J) \tag{8.112}$$

$\downarrow$ $\phi : 2\pi$의 period를 갖는다

$$H = H(I)\theta\text{: } 2\pi\text{의 period를 갖는다.} \tag{8.113}$$

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)}(\theta, I) + \varepsilon^2\phi^{(2)}(\theta, I) \tag{8.114}$$

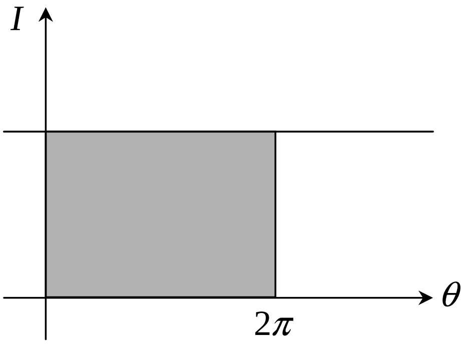
$$J = J^{(0)} + \varepsilon J^{(1)}(\theta, I) + \varepsilon^2 J^{(2)}(\theta, I) \tag{8.115}$$

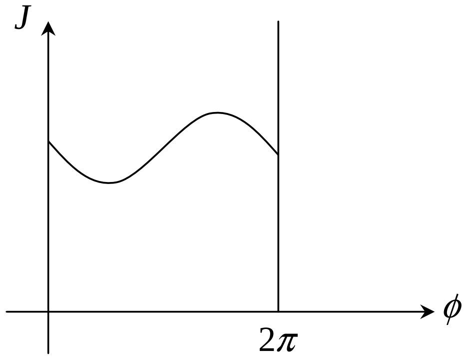
$$\phi^{(0)} = 0, \quad J^{(0)} = I \tag{8.116}$$

$$\theta : 0 \rightarrow 2\pi, \quad \phi : 0 \rightarrow 2\pi \tag{8.117}$$

J는 ϕ의 2π-periodic function

$$\frac{1}{2\pi}\int_0^{2\pi} J(\psi)d\psi = \frac{1}{2\pi}\int_0^{2\pi} I d\theta = I : \text{면적보존} \tag{8.118}$$





$$
\begin{aligned}
d\phi &= d\theta[1+\varepsilon\frac{\partial\phi^{(1)}}{\partial\theta}+\varepsilon^2\frac{\partial\phi^{(2)}}{\partial\theta}] &(8.119)\\
I &= \frac{1}{2\pi}\int_0^{2\pi}[I+\varepsilon J^{(1)}+\varepsilon^2 J^{(2)}][1+\varepsilon\frac{\partial\phi^{(1)}}{\partial\theta}+\varepsilon^2\frac{\partial\phi^{(2)}}{\partial\theta}]d\theta &(8.120)\\
I &= I+\frac{1}{2\pi}\varepsilon\int_0^{2\pi}J^{(1)}d\theta+\frac{1}{2\pi}\varepsilon^2\int_0^{2\pi}[J^{(2)}+J^{(1)}\frac{\partial\phi^{(1)}}{\partial\theta}]d\theta &(8.121)\\
&\quad +O(\varepsilon^3)
\end{aligned}
$$

$$\int[J^{(2)}+J^{(1)}\frac{\partial\phi^{(1)}}{\partial\theta}]d\theta=0 \tag{8.122}$$

$$\begin{vmatrix} 1+\varepsilon\frac{\partial\phi^{(1)}}{\partial\theta}+\varepsilon^2\frac{\partial\phi^{(2)}}{\partial\theta} & \varepsilon\frac{\partial\phi^{(1)}}{\partial I} \\ \varepsilon\frac{\partial J^{(1)}}{\partial\theta} & 1+\varepsilon\frac{\partial J^{(1)}}{\partial I}+\varepsilon^2\frac{\partial J^{(2)}}{\partial I}\end{vmatrix}=1 \tag{8.123}$$

$$\varepsilon[\frac{\partial\phi^{(1)}}{\partial\theta}+\frac{\partial J^{(1)}}{\partial I}]=0 \tag{8.124}$$

$$\varepsilon[\frac{\partial\phi^{(2)}}{\partial\theta}+\frac{\partial J^{(2)}}{\partial I}+\frac{\partial\phi^{(1)}}{\partial\theta}\frac{\partial J^{(1)}}{\partial I}-\frac{\partial\phi^{(1)}}{\partial I}\frac{\partial J^{(1)}}{\partial\theta}]=0 \tag{8.125}$$

$$\frac{\partial\phi^{(2)}}{\partial\theta}+\frac{\partial J^{(2)}}{\partial I}+\frac{\partial(\phi^{(1)},J^{(1)})}{\partial(I,\theta)}=0 \tag{8.126}$$

$$
\begin{aligned}
H &= H_0(q,p)+\varepsilon H_1(q,p) &(8.127)\\
&\downarrow \phi, J\\
H &= H_0(J)+\varepsilon H_1(\phi,J) &(8.128)\\
H &= H_0(I+\varepsilon J^{(1)}+\varepsilon^2 J^{(2)})
\end{aligned}
$$

$$+\varepsilon H_1(\theta+\varepsilon\phi^{(1)}, I+\varepsilon J^{(1)}) + O(\varepsilon^3) \tag{8.129}$$

$$\begin{aligned} H &= H_0(I) + \frac{\partial H_0}{\partial I}(\varepsilon J^{(1)} + \varepsilon^2 J^{(2)}) \\ &+\frac{1}{2}\frac{\partial^2 H_0}{\partial I^2} + \varepsilon H_1(\theta, I) \\ &+\varepsilon^2[\frac{\partial H_1}{\partial \theta}\phi^{(1)} + \frac{\partial H_1}{\partial I}J^{(1)}] \end{aligned} \tag{8.130}$$

$$K_2 = \frac{\partial H_0}{\partial I}J^{(2)} + \frac{1}{2}\frac{\partial^2 H_2}{\partial I^2}[J^{(1)}]^2 + \frac{\partial H_1}{\partial \theta}\phi^{(1)} + \frac{\partial H_1}{\partial I}J^{(1)} \tag{8.131}$$

$$\begin{aligned} K_2 &= \frac{1}{2\pi}\int_0^{2\pi}[\omega_0(I)J^{(2)} + \frac{1}{2}\frac{\partial \omega_0(I)}{\partial I}[J^{(1)}]^2 \\ &+\frac{\partial H_1}{\partial \theta}\phi^{(1)} + \frac{\partial H_1}{\partial I}J^{(1)}]d\theta \end{aligned} \tag{8.132}$$

$$\omega_0(I)J^{(2)} = 0, \frac{\partial H_1}{\partial \theta}\phi^{(1)} = 0 \tag{8.133}$$

$$J^{(1)} = \frac{K_1 - H_1}{\omega_0(I)} \to \frac{\partial J^{(1)}}{\partial \theta} = -\frac{1}{\omega_0}\frac{\partial H_1}{\partial \theta} \tag{8.134}$$

$$\int_0^{2\pi}\omega_0 J^{(2)} = -\omega_0\int_0^{2\pi}J^{(1)}\frac{\partial \phi^{(1)}}{\partial \theta} = \omega_0\int_0^{2\pi}\phi^{(1)}\frac{\partial J^{(1)}}{\partial \theta} = -\int_0^{2\pi}\phi^{(1)}\frac{\partial H_1}{\partial \theta} \tag{8.135}$$

$$\frac{1}{2\pi}\int(K_1 - H_1)^2 d\theta = \frac{1}{2\pi}\int(K_1^2 + H_1^2 - 2K_1H_1)d\theta \tag{8.136}$$

$$= \frac{1}{2\pi}\int(H_1^2 - K_1^2)d\theta \tag{8.137}$$

$$K_2(I) = \frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2}\frac{\partial \omega_0}{\partial I}\frac{H_1^2 - K_1^2}{\omega_0^2} + \frac{K_1 - H_1}{\omega_0}\frac{\partial H_1}{\partial I}]d\theta \tag{8.138}$$

$$\int K_1\frac{\partial H_1}{\partial I} = K_1\frac{\partial}{\partial I}\int H_1 = K_1\frac{\partial K_1}{\partial I} = \frac{1}{2}\frac{\partial K_1^2}{\partial I} \tag{8.139}$$

$$\int H_1\frac{\partial H_1}{\partial I} = \frac{1}{2}\int\frac{\partial H_1^2}{\partial I} \tag{8.140}$$

$$= \frac{1}{4\pi}\int_0^{2\pi}[\frac{\partial}{\partial I}(\frac{1}{\omega_0})(K_1^2 - H_1^2) + \frac{1}{\omega_0}\frac{\partial}{\partial I}(K_1^2 - H_1^2)] \tag{8.141}$$

$$\therefore K_2 = \frac{1}{4\pi}\frac{\partial}{\partial I}\int_0^{2\pi} \frac{K_1^2 - H_1^2}{\omega_0} d\theta \tag{8.142}$$

Eq. (8.142) → (8.131) $J^{(2)}$를 구함. Eq. (8.126) →

$$\frac{\partial \phi^{(2)}}{\partial \theta} = -\frac{\partial J^{(2)}}{\partial I} - \frac{\partial(\phi^{(1)}, J^{(1)})}{\partial(I, \theta)} \tag{8.143}$$

$\int_0^{2\pi} \phi d\theta = 0$이 되게 ϕ를 결정

$$\int_0^{2\pi} \phi^{(2)}(\theta, I) d\theta = 0 \tag{8.144}$$