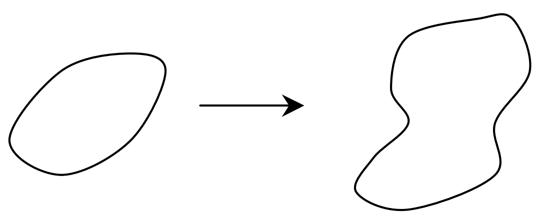
Chapter 2

Linear Transformations of the Plane

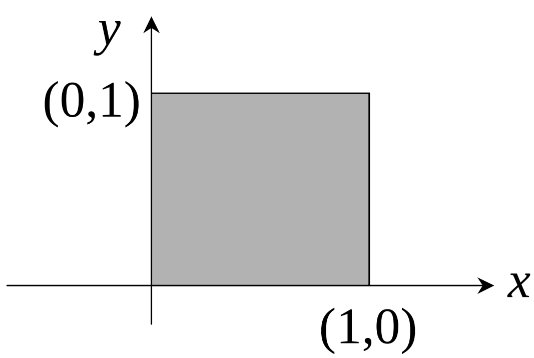
$$\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.1}$$

선형본뜨기: 직선 → 직선

닫힌그림꼴 → 닫힌 그림꼴



단위 네모꼴의 time-evolution을 보자.



본래 네모꼴과 본뜬 그림꼴의 면적이 같을 경우와 다를 경우 두가지를 생각하자.

면적이 보존되는 본뜨기: Hamiltonian 계

면적이 보존 안되는 본뜨기: contracting or expanding

$$DetA = ad - bc = \delta, \quad \delta \neq 0 : \text{가정} \rightarrow A : \text{nonsingular} \tag{2.2}$$

$|\delta| = 1$: 면적보존

$|\delta| \neq 1$: 면적보존 (X)

$$dX \cdot dY = \frac{\partial(X,Y)}{\partial(x,y)} dxdy \tag{2.3}$$

2.1 Area-preserving (면적보존) 변환

$$|A - \lambda I| = 0 \tag{2.4}$$

$\delta = 1$인 경우

$$\lambda^2 - TrA\lambda + 1 = 0 \tag{2.5}$$

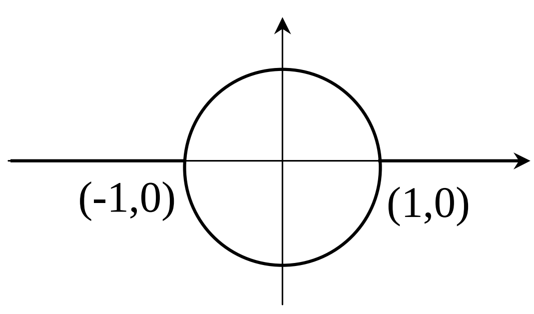
$$TrA = \tau = a + d \tag{2.6}$$

$$\lambda_{1,2} = \frac{\tau}{2} \pm \sqrt{(\frac{\tau}{2})^2 - 1} \tag{2.7}$$

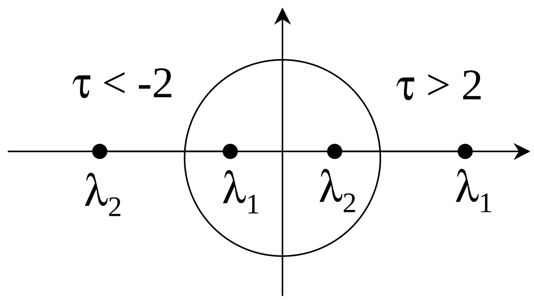
(1) $|\tau| > 2$ 이면 $\lambda_{1,2}$: 실수

(2) $|\tau| < 2$ 이면 $|\lambda_{1,2}| = 1$, 서로 공액

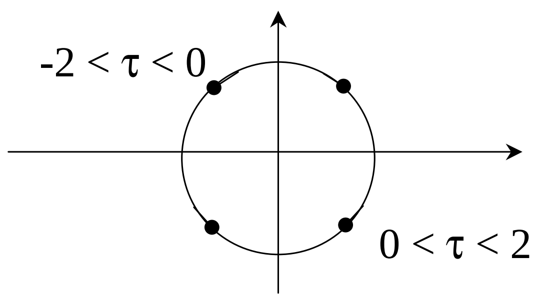
(3) $|\tau| = 2$ 이면 $\lambda_{1,2} = 1$



(1) $\lambda_1 \cdot \lambda_2 = 1$, $\lambda_1 > \lambda_2$라 하자



(2) $\lambda_1 = \alpha + i\omega = e^{i\theta}$, $\lambda_2 = \alpha - i\omega = e^{-i\theta} \rightarrow \lambda_1 \cdot \lambda_2 = \alpha^2 + \omega^2 = 1.$

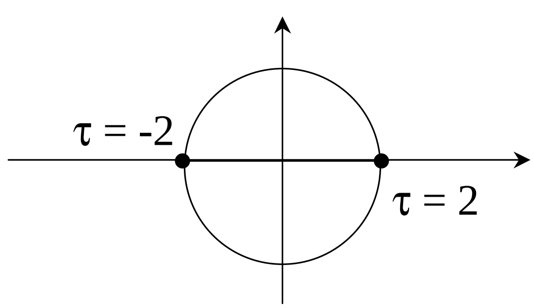


α, ω, 그리고 θ는 모두 실수이고 ω의 부호를 A의 원소 c의 부호와 일치시키는 관례를 취한다.

$$\text{sign}\ \omega = \text{sign}\ c, \quad 0 \le \theta \le \pi \tag{2.8}$$

(3) $|\tau| = 2$

$$\tau = 2 \ \text{ or } \ \tau = -2 \tag{2.9}$$



고유치의 3종이 본뜨기 A에도 판이한 구별을 준다. 이것을 보기 위해서 A를 어떤 비특이 2×2 행렬 M의 닮은 변환을 통한 표준형 B를 만들어서 고찰한다.

$$B = MAM^{-1} \tag{2.10}$$

2-dim.에선 Tr와 Det 값이 eigenvalues를 결정.

$$Det|A| = Det|B|, \quad TrA = TrB \tag{2.11}$$

$\therefore$ eigenvalue도 보존

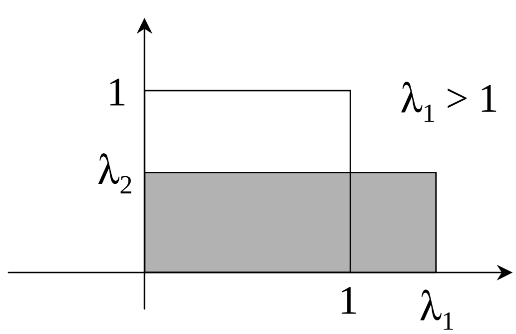
- 제 1 종

$$\text{right eigenvector: } \begin{pmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{pmatrix} \tag{2.12}$$

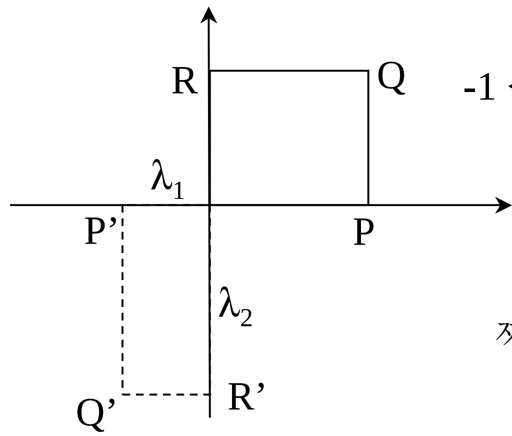
$$M = \begin{pmatrix} c & \lambda_1 - a \\ c & \lambda_2 - a \end{pmatrix} \text{A의 left eigenvector} \tag{2.13}$$

$$\downarrow \quad M^{-1} = \frac{1}{c(\lambda_2 - \lambda_1)} \begin{pmatrix} \lambda_2 - a & a - \lambda_1 \\ -c & c \end{pmatrix} \tag{2.14}$$

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.15}$$



찌브림 (squeeze)



$-1 < \lambda_1 < 0$

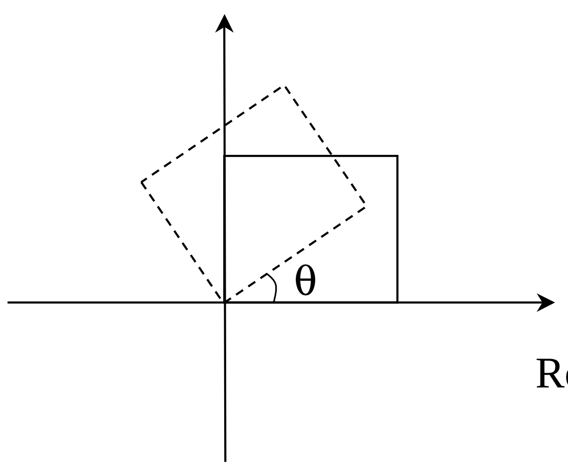
찌브림 + 반전(reflection)

- 제 2 종

$$M = \begin{pmatrix} c & \alpha - a \\ 0 & \omega \end{pmatrix} \tag{2.16}$$

$$M^{-1} = \frac{1}{\omega c}\begin{pmatrix} \omega & a - \alpha \\ 0 & c \end{pmatrix} \tag{2.17}$$

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} \tag{2.18}$$



Rotation

일반적인 matrix의 eigenvectors는 orhtogonal (X)

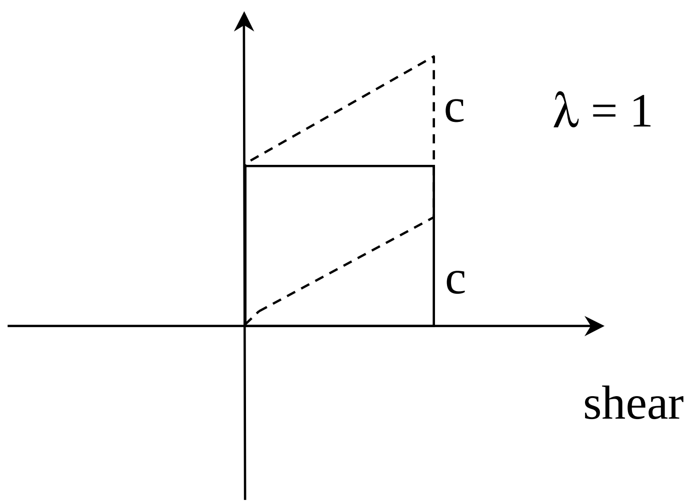
$\therefore$ degeneracy 존재하면 diagonalize (X)

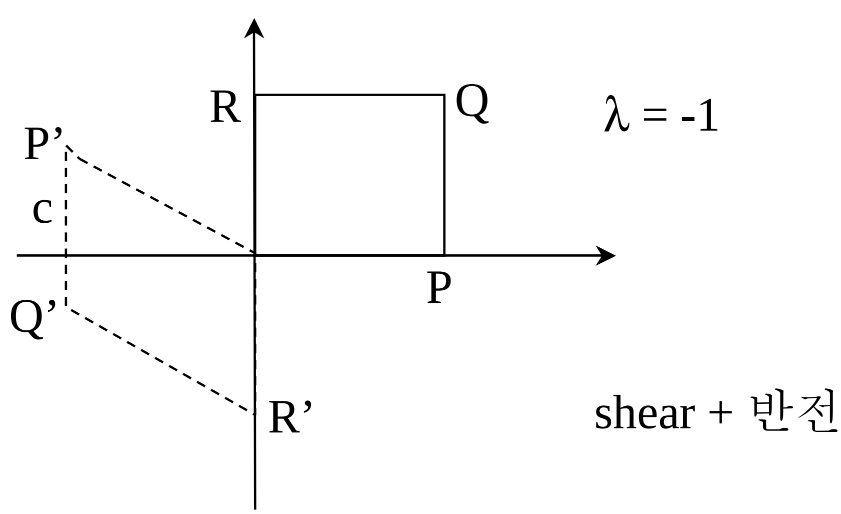
- 제 3 종

$$M = \begin{pmatrix} a-d & 2b \\ 2c & 0 \end{pmatrix} \tag{2.19}$$

$$\downarrow \quad M^{-1} = \frac{1}{4bc}\begin{pmatrix} 0 & 2b \\ 2c & d-a \end{pmatrix} \tag{2.20}$$

$$B = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}, \quad \lambda = \pm 1 \tag{2.21}$$





2.2 흩어지기 변환 (dissipative transformation): $|\delta| < 1$

$$
\begin{aligned}
A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} && (2.22)\\
\delta &= DetA = ab - bc = const. = \lambda_1 \cdot \lambda_2 \ \ |\delta| < 1 && (2.23)\\
\tau &= TrA = a + d = \lambda_1 + \lambda_2 && (2.24)
\end{aligned}
$$

$$\lambda^2 - \tau\lambda + \delta = 0 \tag{2.25}$$

$$\downarrow$$

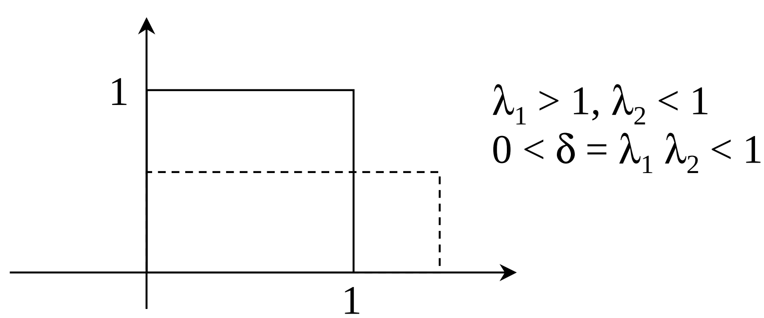
$$\lambda = \tfrac{\tau}{2} \pm \left[\tfrac{\tau^2}{4} - \delta\right]^{1/2} \tag{2.26}$$

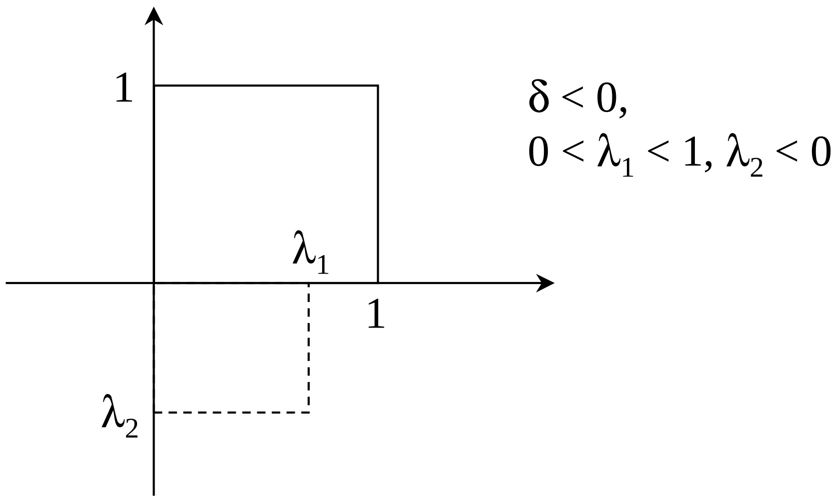
- 1종: $\tau^2 > 4\delta$

Two distinct real eigenvalues, δ can be $+ve$ or $= ve$.

$$\delta > 0 \rightarrow A = \delta^{1/2} A_{AP}, A_{AP} = \begin{pmatrix} a/\delta^{1/2} & b/\delta^{1/2} \\ c/\delta^{1/2} & d/\delta^{1/2} \end{pmatrix} \tag{2.27}$$

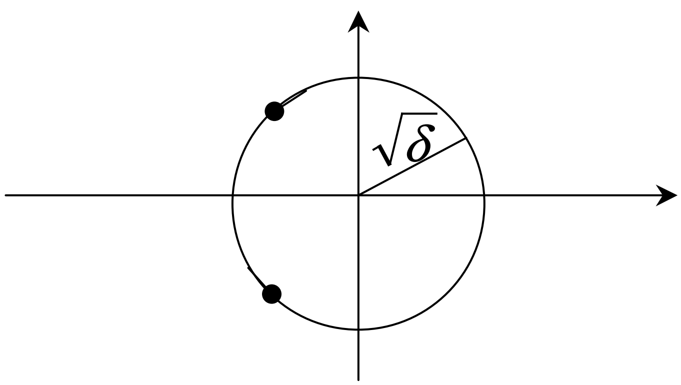
$$
\begin{aligned}
B &= MAM^{-1} = \delta^{1/2} M A_{AP} M^{-1} = \delta^{1/2} B_{AP} && (2.28)\\
B &= \begin{pmatrix} \lambda_1 & 0 \\ 0\lambda_2 & \end{pmatrix} && (2.29)
\end{aligned}
$$





- 2종: $\tau^2 < 4\delta$

Complex conjugate eigenvalues



$$\lambda_1 = \alpha + i\omega = \sqrt{\delta}e^{i\theta}, \lambda_2 = \alpha - i\omega = \sqrt{\delta}e^{-i\theta} \tag{2.30}$$

α, ω, θ: real, $0 < \theta < \pi$

$$B = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} = \sqrt{\delta}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.31}$$

Rotation + contraction

- 3종: $\tau^2 = 4\delta$

$$\lambda = \pm\delta^{1/2} \tag{2.32}$$

$$B = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} \tag{2.33}$$

shear + contraction: $\lambda > 0$

shear + contraction + reflection: $\lambda < 0$