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On the Occurrence of Partial Synchronization in Unidirectionally Coupled Maps

Woochang LIM* and Sang-Yoon KIM†
Department of Physics, Kangwon National University, Chunchon 200-701

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We study three unidirectionally coupled one-dimensional unimodal maps by changing the order α $(1 \leq \alpha \leq 2)$ of the local maximum. A fully synchronized chaotic attractor on the diagonal becomes transversely unstable via a blowout bifurcation; then, partial synchronization or complete desynchronization occurs depending on the value of α. For the quadratic case with $\alpha = 2$, a partially synchronized chaotic attractor appears on an invariant plane. However, as the parameter α decreases and passes a threshold value α^*, a transition from partial synchronization to complete desynchronization takes place. Hence, for $1 \leq \alpha < \alpha^*$, a completely desynchronized chaotic attractor occupies a finite three-dimensional volume because the two-cluster state on the invariant plane, born via the blowout bifurcation, is transversely unstable. The mechanism for the occurrence of partial synchronization is discussed through competition between the laminar and the bursting components of the intermittent two-cluster state born at the blowout bifurcation.

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Recently, because of its potential practical applications (*e.g.*, see Ref. [1]), synchronization of coupled chaotic systems has become a topic of great interest. For a sufficiently strong coupling, a complete synchronization of chaotic systems occurs (*i.e.*, all subsystems become synchronized) [2–5]. However, as the coupling parameter decreases and passes a threshold value, the fully synchronized chaotic attractor on the invariant synchronization subspace becomes transversely unstable via a blowout bifurcation [6, 7]. Then, a partial synchronization (or clustering), where some of the subsystems synchronize while others do not, or complete desynchronization may occur for three or more coupled systems [8–13]. Particularly, the partial synchronization has been extensively investigated in globally coupled systems where each subsystem is coupled to all the other subsystems with equal strength [14].

Here, we are interested in the type of asynchronous intermittent attractors born via the blowout bifurcation of the fully synchronized attractor in the unidirectionally coupled case where one master subsystem drives all the other slave subsystems. Particularly, we are concerned about whether the asynchronous intermittent attractor born at the blowout bifurcation is partially synchronized or completely desynchronized. Examples of partially synchronized attractors were reported in three unidirectionally coupled logistic and Hénon maps [8] while examples of completely desynchronized attractors were given in three unidirectionally coupled tent and circle maps [9]. These results show that partial synchronization depends on the type of base map constituting the coupled system. However, the mechanism for the occurrence of partial synchronization remains unclear.

In this paper, we study three unidirectionally coupled 1D unimodal maps with local maxima of order α $(1 \leq \alpha \leq 2)$:

$$\begin{aligned} x_{n+1} &= f(x_n), \\ y_{n+1} &= f(y_n) + c[f(x_n) - f(y_n)], \\ z_{n+1} &= f(z_n) + c[f(x_n) - f(z_n)], \end{aligned} \qquad (1)$$

where $f(x) = 1 - a|x|^\alpha$. The first drive subsystem with state variable x acts on the second and the third response subsystems with state variables y and z. The base map f for the case of $\alpha = 2$ is just the quadratic map, corresponding to the logistic map studied in Ref. [8], while the base map for the case of $\alpha = 1$ corresponds to the tent map studied in Ref. [9]. We choose a value of the nonlinearity parameter a at which a chaotic attractor exists in f and investigate the occurrence of partial synchronization by varying α from 2 to 1.

For $\alpha = 2$, a fully synchronized attractor exists on the invariant diagonal in the strong-coupling case, as shown in Figs. 1(a)-1(b) for $a = 1.95$ and $c = 0.5$. The longitudinal stability of a synchronized trajectory $\{x_n^* (= y_n^* = z_n^*)\}$ on the attractor against the perturbation along the diagonal (*i.e.*, synchronous perturbation)

*E-mail: wclim@kwnu.kangwon.ac.kr
†E-mail: sykim@kangwon.ac.kr

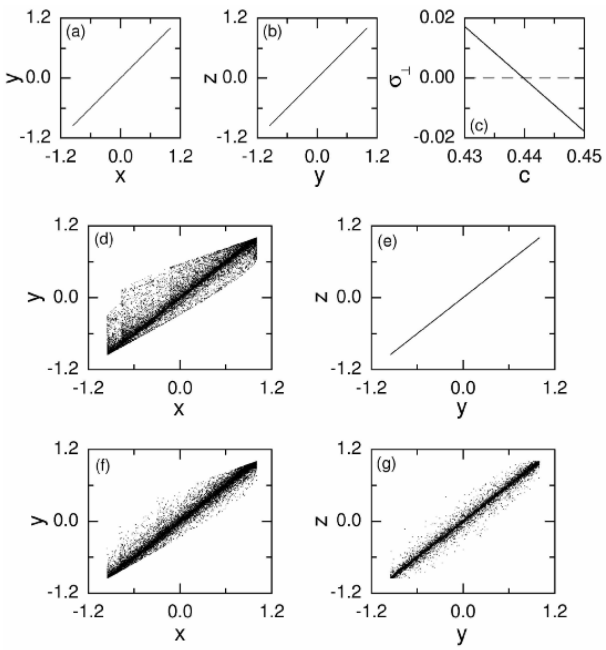

Fig. 1. Partial synchronization and complete desynchronization via a blowout bifurcation of the fully synchronized attractor (a)-(b) A fully synchronized attractor on the diagonal for $a = 1.95$ and $c = 0.5$ in the quadratic case of $\alpha = 2$. (c) Plot of $\sigma_\perp$ (transverse Lyapunov exponent of the fully synchronized attractor) versus c for $\alpha = 2$ and $a = 1.95$. The data of $\sigma_\perp$ are represented by a solid line. (d)-(e) A partially synchronized attractor on the invariant Π_{23} plane for $a = 1.95$ and $c = 0.4348$ in the case of $\alpha = 2$. (f)-(g) A completely desynchronized attractor, occupying a finite 3D volume, for $a = 1.95$ and $c = 0.4565$ in the case of $\alpha = 1.7$.

is determined by its longitudinal Lyapunov exponent

$$\sigma_{||} = \lim_{n\to\infty} \ln |f'(x_n^*)|, \tag{2}$$

where the prime represents the differentiation of f with respect to x. This longitudinal Lyapunov exponent is just the Lyapunov exponent of the uncoupled map f. For $a = 1.95$, we have $\sigma_{||} = 0.5795$; hence, the attractor is a chaotic one. On the other hand, the transverse stability of the fully synchronized attractor against perturbation across the diagonal (*i.e.*, asynchronous perturbation) is determined by its transverse Lyapunov exponent with a two-fold multiplicity,

$$\sigma_\perp = \ln |1 - c| + \sigma_{||}. \tag{3}$$

A plot of $\sigma_\perp$ versus c is shown in Fig. 1(c). If c is relatively large such that $\sigma_{||} < -\ln |1 - c|$, then the fully synchronized attractor becomes transversely stable because its transverse Lyapunov exponent $\sigma_\perp$ is negative. However, as c decreases and passes a threshold value c^* $(= 0.4398)$, the transverse Lyapunov exponent becomes positive; hence, the fully synchronized attractor becomes transversely unstable. Then, complete synchronization is broken, and a partially synchronized attractor appears via a blowout bifurcation on the invariant Π_{23} $(= \{(x, y, z)|y = z\})$ plane, as shown in Figs. 1(d) and 1(e) for $c = 0.4348$. Note that a typical trajectory on the newly-born attractor exhibits on-off intermittency (*i.e.*, long episodes of nearly synchronous evolution near the main diagonal are occasionally interrupted by short-term bursts) [15–18]. The partially synchronized attractor on the Π_{23} plane is a chaotic one with two longitudinal Lyapunov exponents, $\sigma_{||,1}(= 0.5795)$ and $\sigma_{||,2}(= -0.0047)$, and it is transversely stable against the perturbation across the Π_{23} plane because its transverse Lyapunov exponent $\sigma_\perp(= -0.0047)$ is negative. However, the type of asynchronous attractor, born through the blowout bifurcation, depends on the order α of the local maximum. For the case of $\alpha = 1.7$, a fully synchronized attractor on the main diagonal loses its transverse stability when the coupling parameter passes a threshold value $c^*(= 0.4615)$; hence, complete synchronization is broken. Then, a completely desynchronized chaotic attractor with one positive Lyapunov exponent $\sigma_1(= 0.6201)$, occupying a finite 3D volume, appears, as shown in Fig. 1(f)-1(g) for $a = 1.95$ and $c = 0.4565$. This complete desynchronization is in contrast to the partial synchronization for the case of $\alpha = 2$. Such a complete desynchronization occurs because the two-cluster state on the Π_{23} plane, born via a blowout bifurcation, becomes transversely unstable, as will be shown below.

A two-cluster state appears on the invariant Π_{23} plane through a blowout bifurcation when the fully synchronized attractor on the diagonal becomes transversely unstable. The dynamics of this two-cluster state, satisfying $x_n \equiv X_n^*$ and $y_n = z_n \equiv Y_n^*$, is governed by a reduced 2D map,

$$\begin{aligned} X_{n+1}^* &= f(X_n^*), \\ Y_{n+1}^* &= f(Y_n^*) + c[f(X_n^*) - f(Y_n^*)]. \end{aligned} \tag{4}$$

For the accuracy of the numerical calculations [19], we introduce new coordinates u and v such that

$$u = \frac{X^* + Y^*}{2}, \quad v = \frac{X^* - Y^*}{2}. \tag{5}$$

Under the coordinate change, the invariant diagonal is transformed into a new invariant line $v = 0$. In these new coordinates, the 2D reduced map of Eq. (4) becomes:

$$\begin{aligned} u_n &= \frac{1}{2}(1 + c)f(u_n + v_n) + \frac{1}{2}(1 - c)f(u_n - v_n)], \\ v_n &= \frac{1}{2}(1 - c)[f(u_n + v_n) - f(u_n - v_n)]. \end{aligned} \tag{6}$$

Figures 2(a)-2(b) show the two-cluster states in the $u-v$ plane, born via blowout bifurcations, for $\alpha = 2.0$ and 1.7, respectively. Both the two-cluster states are chaotic attractors in the reduced 2D map (*i.e.*, they are chaotic attractors in the restricted Π_{23} plane). However, their transverse stability against perturbation across the invariant Π_{23} plane in the whole 3D space depends on the value of α. We numerically follow a typical trajectory in the two-cluster state until its length L becomes 10^8;

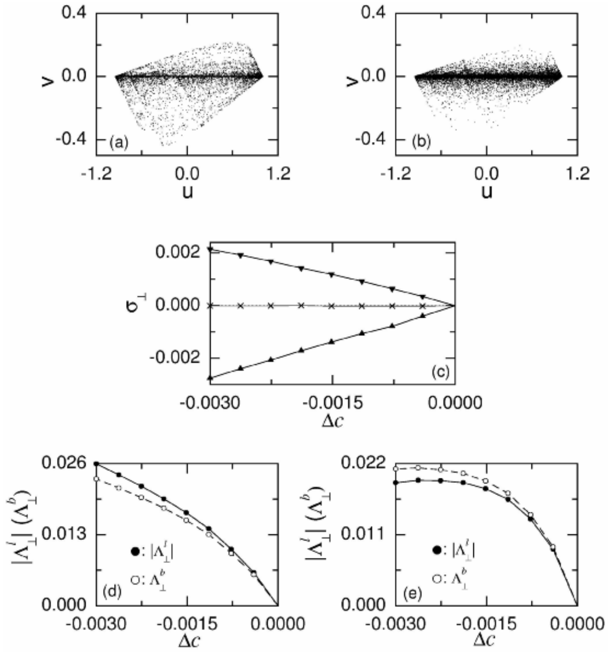

Fig. 2. Transverse stability of two-clusters states born at the blowout bifurcation. (a) Transversely stable two-cluster state for $a = 1.95$ and $\Delta c(= c - c^*) = -\ 0.003$ ($c^* = 0.4398$) in the case of $\alpha = 2$. (b) Transversely unstable two-cluster state for $a = 1.95$ and $\Delta c(= c - c^*) = -\ 0.003$ ($c^* = 0.4615$) in the case of $\alpha = 1.7$. (c) Plot of $\sigma_\perp$ (transverse Lyapunov exponent of the two-cluster state) versus Δc for $a = 1.95$. The data of $\sigma_\perp$ for $\alpha = 2.0$, 1.884, and 1.7 are represented by the up triangles, crosses, and down triangles, respectively. (d)-(e) Plots of $|\Lambda_\perp^l|$ and $\Lambda_\perp^b$ (weighted transverse Lyapunov exponents of the laminar and the bursting components in the two-cluster state, respectively) versus Δc for $d^* = 10^{-4}$.

then, the transverse Lyapunov exponent for the trajectory segment with length L is given by

$$\sigma_\perp = \frac{1}{L} \sum_{n=0}^{L-1} \ln |(1-c)\, f'(u_n - v_n)|. \tag{7}$$

A plot of $\sigma_\perp$ versus Δc $(= c - c^*)$ is given in Fig. 2(c), where c^* is the blowout bifurcation point of the fully synchronized attractor. For the quadratic case of $\alpha = 2$, the two-cluster state is transversely stable because its transverse Lyapunov exponent $\sigma_\perp$ is negative; hence, partial synchronization occurs on the Π_{23} plane via a blowout bifurcation (*i.e.*, a transversely stable chaotic attractor exists on the Π_{23} plane.). On the other hand, as α is decreased from 2, the value of $\sigma_\perp$ increases, eventually it becomes zero for a threshold value $\alpha^*(\simeq 1.884)$, and then it becomes positive. Hence, for $\alpha < \alpha^*$, complete desynchronization takes place through a blowout bifurcation (*i.e.*, a completely desynchronized 3D attractor appears) because the two-cluster state on the Π_{23} plane becomes transversely unstable. As examples for $\Delta c = -\ 0.003$, see Figs. 2(a) and 2(b) that show the transversely stable ($\sigma_\perp = -\ 0.0027$) and unstable ($\sigma_\perp = 0.0021$) two-cluster states for $\alpha = 2.0$ and 1.7, respectively.

Finally, we discuss the mechanism for the transition from partial synchronization to complete desynchronization by varying the order parameter α. A typical trajectory in the two-cluster state, exhibiting on-off intermittency, may be decomposed into laminar (*i.e.*, nearly synchronous) and bursting components. We use a small quantity d^* for the threshold value of the magnitude of the transverse variable d $(= |v|)$ such that for $d < d^*$, the trajectory is considered to be in the laminar (off) state and for $d > d^*$, it is considered to be in the bursting (on) state. Then, the transverse Lyapunov exponent $\sigma_\perp$ of a two-cluster state (see Eq. (7) for the transverse Lyapunov exponent for a trajectory segment) can be given by the sum of the two weighted transverse Lyapunov exponents for the laminar and the bursting components, $\Lambda_\perp^l$ and $\Lambda_\perp^b$:

$$\sigma_\perp = \Lambda_\perp^l + \Lambda_\perp^b \tag{8.1}$$
$$= \Lambda_\perp^b - |\Lambda_\perp^l|, \tag{8.2}$$

where the laminar component always has a negative weighted transverse Lyapunov exponent ($\Lambda_\perp^l < 0$). Here, the weighted transverse Lyapunov exponent $\Lambda_\perp^i$ for each component ($i = l, b$) is given by the product of the fraction, μ_i, of time spent in the i state and its transverse Lyapunov exponent $\sigma_\perp^i$; *i.e.*,

$$\Lambda_\perp^i = \mu_i \sigma_\perp^i;\ \ \mu_i = \frac{L^i}{L},\ \ \sigma_\perp^i = \frac{1}{L^i} {\sum_{n \in\, i\, \mathrm{state}}}' \ln |(1-c)\, f'(u_n - v_n)| \ \ (i = l, b), \tag{9}$$

where L^i is the time spent in the i state for a trajectory segment of length L and the primed summation is performed in each i state. As can be seen in Eq. (8.2), the sign of $\sigma_\perp$ is determined through competition between the laminar and the bursting components. Hence, when the "strength" [*i.e.*, the magnitude of the weighted transverse Lyapunov exponent ($|\Lambda_\perp^l|$)] of the laminar component is larger (smaller) than that (*i.e.*, $\Lambda_\perp^b$) of the bursting component, partial synchronization (complete desynchronization) occurs. Figures 2(d) and 2(e) show the weighted transverse Lyapunov exponents of the laminar and the bursting components for $\alpha = 2.0$ and 1.7, respectively, when $d^* = 10^{-4}$. For the case of $\alpha = 2.0$, partial synchronization occurs on the invariant Π_{23} plane because the laminar component is dominant (*i.e.*, $|\Lambda_\perp^l| > \Lambda_\perp^b$). On the other hand, complete desynchronization takes place in the case of $\alpha = 1.7$ because the bursting component is dominant (*i.e.*, $|\Lambda_\perp^l| < \Lambda_\perp^b$).

In summary, we have investigated the occurrence of partial synchronization via blowout bifurcations of the fully synchronized attractor in three unidirectionally coupled 1D maps by varying the order parameter α of the local maximum. For the quadratic case of $\alpha = 2$, partial synchronization has been found to occur on the invariant Π_{23} plane. However, as α is decreased from 2 and passes a threshold value α^* ($\simeq 1.884$), a transition from partial synchronization to complete desynchronization occurs. Hence, for $\alpha < \alpha^*$ complete desynchronization has been found to take place. This transition can be understood through competition between the laminar and the bursting components of the two-cluster state on the Π_{23} plane, born at the blowout bifurcation. When the laminar (bursting) component is dominant, partial synchronization (complete desynchronization) occurs.

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[19] When the magnitude of a transverse variable d of a typical trajectory in the two-cluster state, representing the deviation from the invariant synchronization line, is less than a threshold value $\tilde{d}$, the computed trajectory falls into an exactly synchronous state due to a finite precision. In the system of coordinates X^* and Y^*, the order of magnitude of the threshold value $\tilde{d}$ for d ($= |X^* - Y^*|$) is about 10^{-15}, except for the region near the origin, because the double-precision values of X^* and Y^* have about 15 decimal places of precision. On the other hand, in the system of u and v, the order of magnitude of the threshold value $\tilde{d}$ for $d (= |v|)$ is about 2.2×10^{-308}, which is a threshold value for the numerical underflow in the double-precision calculation. Hence, in the system of u and v, we can follow a trajectory until its length becomes sufficiently long to calculate the Lyapunov exponents of the two-cluster state.