On the Occurrence of Partial Synchronization in
Unidirectionally Coupled Maps

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We study three unidirectionally coupled one-dimensional unimodal maps by changing the order
α (1 ≤ α ≤ 2) of the local maximum. A fully synchronized chaotic attractor on the diagonal
crosses transversely unstable via a blowout bifurcation; then, partial synchronization or complete
desynchronization occurs depending on the value of α. For the quadratic case with α = 2, a par-
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decreases and passes a threshold value α∗, a transition from partial synchronization to complete
desynchronization takes place. Hence, for 1 ≤ α < α∗, a completely desynchronized chaotic at-
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is determined by its longitudinal Lyapunov exponent
\[\sigma|| = \lim_{n \to \infty} \ln|f'(x^*_n)|,\] (2)
where the prime represents the differentiation of \(f\) with respect to \(x\).
This longitudinal Lyapunov exponent is just the Lyapunov exponent of the uncoupled map \(f\).
For \(\alpha = 1.95\), we have \(\sigma|| = 0.5795\); hence, the attractor is a chaotic one.
On the other hand, the transverse stability of the fully synchronized attractor against perturbation across the diagonal (i.e., asynchronous perturbation) is determined by its transverse Lyapunov exponent with a two-fold multiplicity,
\[\sigma_\perp = \ln|1 - c| + \sigma||.\] (3)
A plot of \(\sigma_\perp\) versus \(c\) is shown in Fig. 1(c). If \(c\) is relatively large such that \(\sigma|| < -\ln|1 - c|\), then the fully synchronized attractor becomes transversely stable because its transverse Lyapunov exponent \(\sigma_\perp\) is negative. However, as \(c\) decreases and passes a threshold value \(c^* (= 0.4398)\), the transverse Lyapunov exponent becomes positive; hence, the fully synchronized attractor becomes transversely unstable. Then, complete synchronization is broken, and a partially synchronized attractor appears via a blowout bifurcation on the invariant \(\Pi_{23} (= \{(x, y, z)|y = z\})\) plane, as shown in Figs. 1(d) and 1(e) for \(c = 0.4348\). Note that a typical trajectory on the newly-born attractor exhibits on-off intermittency (i.e., long episodes of nearly synchronous evolution near the main diagonal are occasionally interrupted by short-term bursts) [15–18].
The partially synchronized attractor on the \(\Pi_{23}\) plane is a chaotic one with two longitudinal Lyapunov exponents, \(\sigma||,1 (= 0.5795)\) and \(\sigma||,2 (= -0.0047)\), and it is transversely stable against the perturbation across the \(\Pi_{23}\) plane because its transverse Lyapunov exponent \(\sigma_\perp (= -0.0047)\) is negative. However, the type of asynchronous attractor, born through the blowout bifurcation, depends on the order \(\alpha\) of the local maximum. For the case of \(\alpha = 1.7\), a fully synchronized attractor on the main diagonal loses its transverse stability when the coupling parameter passes a threshold value \(c^* (= 0.4615)\); hence, complete synchronization is broken. Then, a completely desynchronized chaotic attractor with one positive Lyapunov exponent \(\sigma_1 (= 0.6201)\), occupying a finite 3D volume, appears, as shown in Fig. 1(f)-1(g) for \(\alpha = 1.95\) and \(c = 0.4565\). This complete desynchronization is in contrast to the partial synchronization for the case of \(\alpha = 2\). Such a complete desynchronization occurs because the two-cluster state on the \(\Pi_{23}\) plane, born via a blowout bifurcation, becomes transversely unstable, as will be shown below.
A two-cluster state appears on the invariant \(\Pi_{23}\) plane through a blowout bifurcation when the fully synchronized attractor on the diagonal becomes transversely unstable. The dynamics of this two-cluster state, satisfying \(x^*_n \equiv X^*_n\) and \(y^*_n \equiv z^*_n \equiv Y^*_n\), is governed by a reduced 2D map,
\[\begin{align*}
X^*_{n+1} &= f(X^*_n), \\
Y^*_{n+1} &= f(Y^*_n) + c[f(X^*_n) - f(Y^*_n)].
\end{align*}\] (4)
For the accuracy of the numerical calculations [19], we introduce new coordinates \(u\) and \(v\) such that
\[\begin{align*}
u &= \frac{x^* + y^*}{2}, \\
v &= \frac{x^* - y^*}{2}.
\end{align*}\] (5)
Under the coordinate change, the invariant diagonal is transformed into a new invariant line \(v = 0\). In these new coordinates, the 2D reduced map of Eq. (4) becomes:
\[\begin{align*}
u_n &= \frac{1}{2}(1 + c)f(u_n + v_n) + \frac{1}{2}(1 - c)f(u_n - v_n), \\
v_n &= \frac{1}{2}(1 - c)[f(u_n + v_n) - f(u_n - v_n)].
\end{align*}\] (6)
Figures 2(a)-2(b) show the two-cluster states in the \(u-v\) plane, born via blowout bifurcations, for \(\alpha = 2.0\) and 1.7, respectively. Both the two-cluster states are chaotic attractors in the reduced 2D map (i.e., they are chaotic attractors in the restricted \(\Pi_{23}\) plane). However, their transverse stability against perturbation across the invariant \(\Pi_{23}\) plane in the whole 3D space depends on the value of \(\alpha\). We numerically follow a typical trajectory in the two-cluster state until its length \(L\) becomes \(10^9\);
where $c$ see Figs. 2(a) and 2(b) that show the transversely stable $c$ transversely unstable. As examples for $\Delta \alpha < \alpha$ it becomes positive. Hence, for $\alpha = 2.0$, the trajectory is considered to be in the laminar (off) state and for $d > d^*$, it is considered to be in the bursting (on) state. Then, the transverse Lyapunov exponent $\sigma_\perp$ of a two-cluster state (see Eq. (7) for the transverse Lyapunov exponent for a trajectory segment) can be given by the sum of the two weighted transverse Lyapunov exponents for the laminar and the bursting components, $\Lambda^l_\perp$ and $\Lambda^b_\perp$:

$$\sigma_\perp = \Lambda^l_\perp + \Lambda^b_\perp \quad \text{(8.1)}$$

$$= \Lambda^l_\perp - |\Lambda^l_\perp|, \quad \text{(8.2)}$$

where the laminar component always has a negative weighted transverse Lyapunov exponent ($\Lambda^l_\perp < 0$). Here, the weighted transverse Lyapunov exponent $\Lambda^i_\perp$ for each component ($i = l, b$) is given by the product of the fraction, $\mu_i$, of time spent in the $i$ state and its transverse Lyapunov exponent $\sigma^i_\perp$; i.e.,

$$\Lambda^i_\perp = \mu_i \sigma^i_\perp; \quad \mu_i = \frac{L^i}{L}; \quad \sigma^i_\perp = \frac{1}{L^i} \sum_{n \in i \text{ state}} \ln |(1 - c) f'(u_n - v_n)| \quad (i = l, b), \quad \text{(9)}$$

where $L^i$ is the time spent in the $i$ state for a trajectory segment of length $L$ and the primed summation is performed in each $i$ state. As can be seen in Eq. (8.2), the sign of $\sigma_\perp$ is determined through competition between the laminar and the bursting components. Hence, when the “strength” [i.e., the magnitude of the weighted transverse Lyapunov exponent ($|\Lambda^l_\perp|$) of the laminar component is larger (smaller) than that (i.e., $\Lambda^b_\perp$) of the bursting component, partial synchronization (complete desynchronization) occurs. Figures 2(d) and 2(e) show the weighted transverse Lyapunov exponents of the laminar and the bursting components for $\alpha = 2.0$ and 1.7, respectively, when $d^* = 10^{-4}$. For the case of $\alpha = 2.0$, partial synchronization occurs on the invariant $\Pi_{23}$ plane because the laminar component is dominant (i.e., $|\Lambda^l_\perp| > |\Lambda^b_\perp|$). On the other hand, complete desynchronization takes place in the case of $\alpha = 1.7$ because the bursting component is dominant (i.e., $|\Lambda^l_\perp| < |\Lambda^b_\perp|$).

In summary, we have investigated the occurrence of partial synchronization via blowout bifurcations of the fully synchronized attractor in three unidirectionally coupled 1D maps by varying the order parameter $\alpha$ of the local maximum. For the quadratic case of $\alpha = 2$, partial synchronization has been found to occur on the invariant $\Pi_{23}$ plane. However, as $\alpha$ is decreased from 2

![Fig. 2. Transverse stability of two-clusters states born at the blowout bifurcation. (a) Transversely stable two-cluster state for $a = 1.95$ and $\Delta c = c - c^* = -0.003 (c^* = 0.3898)$ in the case of $\alpha = 2$. (b) Transversely unstable two-cluster state for $a = 1.95$ and $\Delta c = c - c^* = -0.003 (c^* = 0.4615)$ in the case of $\alpha = 1.7$. (c) Plot of $\sigma_\perp$ (transverse Lyapunov exponent of the two-cluster state) versus $\Delta c$ for $a = 1.95$. The data of $\sigma_\perp$ for $\alpha = 2.0$, 1.884, and 1.7 are represented by the up triangles, crosses, and down triangles, respectively. (d)-(e) Plots of $|\Lambda^l_\perp|$ and $\Lambda^l_\perp$ (weighted transverse Lyapunov exponents of the laminar and the bursting components in the two-cluster state, respectively) versus $\Delta c$ for $d^* = 10^{-4}$.](image)
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and passes a threshold value $\alpha^*$ ($\simeq 1.884$), a transition from partial synchronization to complete desynchronization occurs. Hence, for $\alpha < \alpha^*$ complete desynchronization has been found to take place. This transition can be understood through competition between the laminar and the bursting components of the two-cluster state on the $\Pi_{23}$ plane, born at the blowout bifurcation. When the laminar (bursting) component is dominant, partial synchronization (complete desynchronization) occurs.

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REFERENCES


[19] When the magnitude of a transverse variable $d$ of a typical trajectory in the two-cluster state, representing the deviation from the invariant synchronization line, is less than a threshold value $\tilde{d}$, the computed trajectory falls into an exactly synchronous state due to a finite precision. In the system of coordinates $X^*$ and $Y^*$, the order of magnitude of the threshold value $\tilde{d}$ for $d = |X^* - Y^*|$ is about $10^{-15}$, except for the region near the origin, because the double-precision values of $X^*$ and $Y^*$ have about 15 decimal places of precision. On the other hand, in the system of $u$ and $v$, the order of magnitude of the threshold value $\tilde{d}$ for $d = |v|$ is about $2.2 \times 10^{-308}$, which is a threshold value for the numerical underflow in the double-precision calculation. Hence, in the system of $u$ and $v$, we can follow a trajectory until its length becomes sufficiently long to calculate the Lyapunov exponents of the two-cluster state.