Mechanism for the intermittent route to strange nonchaotic attractors

Sang-Yoon Kim,[1,2,*] Woochang Lim,[2] and Edward Ott[1,3]
[1]*Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742*
[2]*Department of Physics, Kangwon National University, Chunchon, Kangwon-Do 200-701, Korea*
[3]*Department of Physics and Department of Electrical and Computing Engineering, University of Maryland, College Park, Maryland 20742*
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Intermittent strange nonchaotic attractors (SNAs) appear typically in quasiperiodically forced period-doubling systems. As a representative model, we consider the quasiperiodically forced logistic map and investigate the mechanism for the intermittent route to SNAs using rational approximations to the quasiperiodic forcing. It is found that a smooth torus transforms into an intermittent SNA via a phase-dependent saddle-node bifurcation when it collides with a new type of ''ring-shaped'' unstable set.

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I. INTRODUCTION

Recently, much attention has been paid to the study of quasiperiodically forced systems because they generically can have strange nonchaotic attractors (SNAs) [1]. SNAs were first described by Grebogi *et al.* [2] and have been extensively investigated both numerically [3–16] and experimentally [17]. SNAs exhibit properties of regular as well as chaotic attractors. Like regular attractors, their dynamics is nonchaotic in the sense that they do not have a positive Lyapunov exponent; such as typical chaotic attractors, they exhibit a fractal phase space structure. Furthermore, SNAs are related to the Anderson localization in the Schrödinger equation with a spatially quasiperiodic potential [18], and they may have a practical application in secure communication [19]. Therefore, dynamical transitions in quasiperiodically forced systems have become a topic of considerable current interest. However, the mechanisms of their appearance, as a system parameter is varied, are much less clear than those of unforced or periodically forced systems.

Here, we are interested in the dynamical transition to SNAs accompanied by intermittent behavior, as reported in Ref. [12]. As a parameter passes a threshold value, a smooth torus abruptly transforms into an intermittent SNA. Near the transition point, the intermittent dynamics on the SNA was characterized in terms of the average interburst time and the Lyapunov exponent. This route to an intermittent SNA is quite general and has been observed in a number of quasiperiodically forced period-doubling maps and flows (e.g., see Refs. [13,14]). It has been suggested [15] that the observed intermittent behavior results through an interaction with an unstable orbit. However, in the previous work, such an unstable orbit was not located, and thus the bifurcation mechanism for the intermittent transition remains unclear.

This paper is organized as follows. In Sec. II, we investigate the underlying mechanism for the intermittent transition in the quasiperiodically forced logistic map which we regard as a representative model for quasiperiodically forced period-doubling systems. Using rational approximations (RAs) to the quasiperiodic forcing, we observe a new type of invariant unstable set, which will be referred to as a ''ring-shaped'' unstable set in accordance with its geometry. When a smooth torus (corresponding to an ordinary quasiperiodic attractor) collides with this ring-shaped unstable set, a transition to an intermittent SNA is found to occur through a ''phase-dependent saddle-node'' bifurcation. In a future paper [20], it will be shown that other dynamical transitions such as interior, boundary, and band-merging crises may also occur through an interaction with the ring-shaped unstable set. Consequently, the ring-shaped unstable sets play a central role for these typical dynamical transitions. We also note that these kinds of dynamical transitions seem to be ''universal,'' in the sense that they occur in typical quasiperiodically forced period-doubling systems such as the quasiperiodically forced Hénon map, ring map, and pendulum [20,21]. Finally, a summary is given in Sec. III.

II. INTERMITTENT TRANSITION TO STRANGE NONCHAOTIC ATTRACTORS

We investigate the mechanism for the intermittent route to SNAs in the quasiperiodically forced logistic map M [5]:

$$M:\begin{cases} x_{n+1}=(a+\varepsilon\cos 2\pi\theta_n)x_n(1-x_n) \\ \theta_{n+1}=\theta_n+\omega \pmod{1}, \end{cases} \tag{1}$$

where $x\in[0,1]$, $\theta\in S^1$, a is the nonlinearity parameter of the logistic map, and ω and ε represent the frequency and amplitude of the quasiperiodic forcing, respectively. We set the frequency to be the reciprocal of the golden mean, $\omega=(\sqrt{5}-1)/2$. The intermittent transition is then investigated using RAs. For the inverse golden mean, its rational approximants are given by the ratios of the Fibonacci numbers, $\omega_k=F_{k-1}/F_k$, where the sequence of $\{F_k\}$ satisfies $F_{k+1}=F_k+F_{k-1}$ with $F_0=0$ and $F_1=1$. Instead of the quasiperiodically forced system, we study an infinite sequence of periodically forced systems with rational driving frequencies ω_k. We assume that the properties of the original system M may be obtained by taking the quasiperiodic limit $k\to\infty$. Using this technique, a transition from a smooth torus to an intermittent SNA is found to occur through a collision with a new type of ring-shaped unstable set.

*Electronic address: sykim@kangwon.ac.kr-

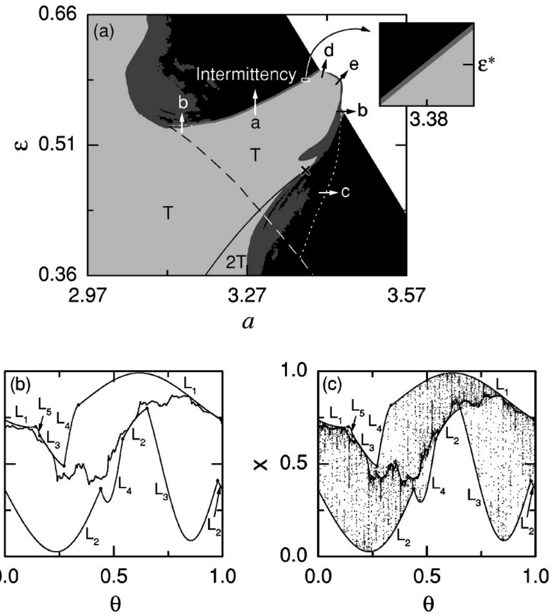

FIG. 1. (a) Phase diagram in the a-ε plane. Regular, chaotic, SNA, and divergence regimes are shown in light gray, black, gray (or dark gray), and white, respectively. To show the region of existence (gray) of the intermittent SNA occurring between T (light gray) and the chaotic attractor region (black), a small box near $(a,\varepsilon)=[3.38,\varepsilon^*(=0.584\,726\,781)]$ is magnified. Through an interaction with the ring-shaped unstable set born when passing the dashed line, typical dynamical transitions such as the intermittency (route a) and the interior (routes b and c; the dotted line is an interior crisis line) and boundary (routes d and e) crises may occur. Here, the torus and the doubled torus are denoted by T and $2T$ and the solid line represents a torus-doubling bifurcation curve whose terminal point is marked with the cross. (b) Smooth torus inside an absorbing area with boundary formed by segments of the critical curves L_k $(k=1,\ldots,5)$ (the dots indicate where these segments connect) for $a=3.38$ and $\varepsilon=0.5847$. (c) SNA filling the absorbing area for $a=3.38$ and $\varepsilon=0.584\,75$. For other details, see the text.

The quasiperiodically forced logistic map M is noninvertible because its Jacobian determinant becomes zero along the critical curve, $L_0=\{x=0.5,\theta\in[0,1)\}$. Critical curves of rank k, $L_k(k=1,2,\ldots)$, are then given by the images of L_0, [i.e., $L_k=M^k(L_0)$]. Segments of these critical curves can be used to define a bounded trapping region of the phase space, called an "absorbing area," inside which, upon entering, trajectories are henceforth confined [22]. It is found that the newly born intermittent SNA fills the absorbing area. Hence, the critical curves determine the global structure of the SNA.

Figure 1(a) shows a phase diagram in the a-ε plane. Each phase is characterized by the Lyapunov exponent σ_x in the x direction as well as the phase sensitivity exponent δ. The exponent δ measures the sensitivity with respect to the phase of the quasiperiodic forcing, and was introduced in Ref. [6] to characterize the strangeness of an attractor of a quasiperiodically driven system. A smooth torus that has a negative Lyapunov exponent without phase sensitivity $(\delta=0)$ exists in the region denoted by T and shown in light gray. Upon crossing the solid line, the smooth torus becomes unstable and bifurcates to a smooth doubled torus in the region denoted by $2T$. Chaotic attractors with positive Lyapunov exponents exist in the region shown in black. Between these regular and chaotic regions, SNAs that have negative Lyapunov exponents with high phase sensitivity $(\delta>0)$ exist in the regions shown in gray and dark gray. Consistent with their positive phase sensitivity exponent δ, these SNAs are observed to have a fractal structure [6]. Here, we restrict our considerations only to the intermittent SNAs that exist in the thin gray region [e.g., see a magnified part in Fig. 1(a)]. (In the dark-gray region, nonintermittent SNAs, born through other mechanisms, such as gradual fractalization [10] and torus collision [5], exist.) This phase diagram is typical for quasiperiodically forced period-doubling systems [12–16,20,21], and its main interesting feature is the existence of the "tongue" of quasiperiodic motion that penetrates into the chaotic region and separates it into upper and lower parts. We also note that this tongue lies near the terminal point (denoted by the cross) of the torus-doubling bifurcation curve. When crossing the upper boundary of the tongue, a smooth torus transforms into an intermittent SNA that exists in the thin gray region. Hereafter, this intermittent route to SNAs will be referred to as route a [see Fig. 1(a)].

As an example, we consider the case $a=3.38$. Figure 1(b) shows a smooth torus with $\sigma_x=-0.059$ for $\varepsilon=0.584\,7$ inside an absorbing area whose boundary is formed by segments of the critical curves L_k $(k=1,\ldots,5)$ (the dots indicate where these segments connect). We also note that the smooth unstable torus $x=0$ and its first preimage $x=1$ form the basin boundary of the smooth torus in the θ-x plane. However, as ε passes a threshold value ε^* $(=0.584\,726\,781)$, a transition to an intermittent SNA occurs. As shown in Fig. 1(c) for $\varepsilon=0.584\,75$, the newly-born intermittent SNA with $\sigma_x=-0.012$ and $\delta=19.5$ appears to fill the absorbing area, and its typical trajectory spends most of its time near the former torus with sporadic large bursts away from it. This intermittent transition may be expected to have occurred through the collision of the smooth attracting torus with an unstable orbit. However, the smooth unstable torus $x=0$ cannot interact with the smooth stable torus because it lies outside the absorbing area. Hence, we search inside the absorbing area for an unstable orbit that might collide with the smooth stable torus.

Using RAs we find a new type of ring-shaped unstable set that causes the intermittent transition through a collision with the smooth torus. When passing the dashed curve in Fig. 1(a), such a ring-shaped unstable set appears via a phase-dependent saddle-node bifurcation. This bifurcation has no counterpart in the unforced case. (The dashed line is numerically obtained for a sufficiently large level $k=10$ of the RAs.) For each RA of level k, a periodically forced logistic map with rational driving frequency ω_k has a periodic or a chaotic attractor that depends on the initial phase θ_0 of the external forcing. Then the union of all attractors for different θ_0 gives the kth approximation to the attractor in the quasiperiodically forced system. As an example, consider the RA of low level $k=6$. As shown in Fig. 2(a) for $a=3.246$ and $\varepsilon=0.446$, the RA to the smooth torus (denoted by a black line), consisting of stable orbits with period F_6 $(=8)$, exists inside an absorbing area bounded by segments of the critical

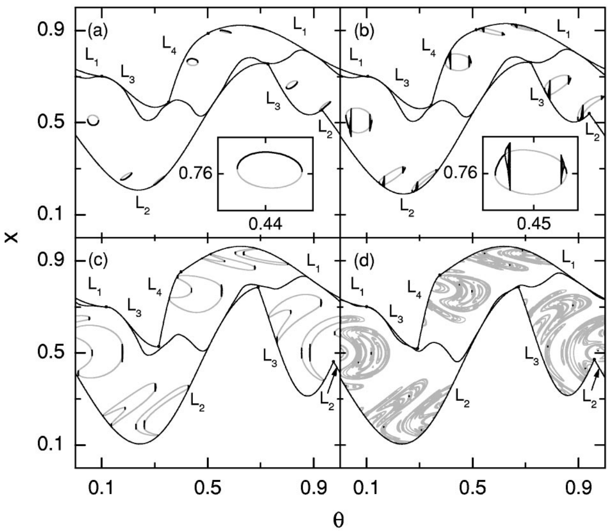

FIG. 2. Smooth torus and ring-shaped unstable set in the RA of level 6 for (a) $a=3.246$ and $\varepsilon=0.446$, (b) $a=3.26$ and $\varepsilon=0.46$, and (c) $a=3.326$ and $\varepsilon=0.526$. (d) Smooth torus and ring-shaped unstable set in the RA of level 8 for $a=3.326$ and $\varepsilon=0.526$. Both the smooth torus (denoted by a black line) and the ring-shaped unstable set (composed of rings) exist inside the absorbing area with boundary formed by portions of the critical curves L_k ($k=1,\ldots,4$) (the dots indicate where these portions connect). For the RA of level k, each ring is composed of the attracting part (shown in black) and the unstable part (shown in gray and consisting of unstable F_k-periodic orbits). As the level k increases, the unstable part becomes dominant, since the attracting part becomes negligibly small. For more details, see the text.

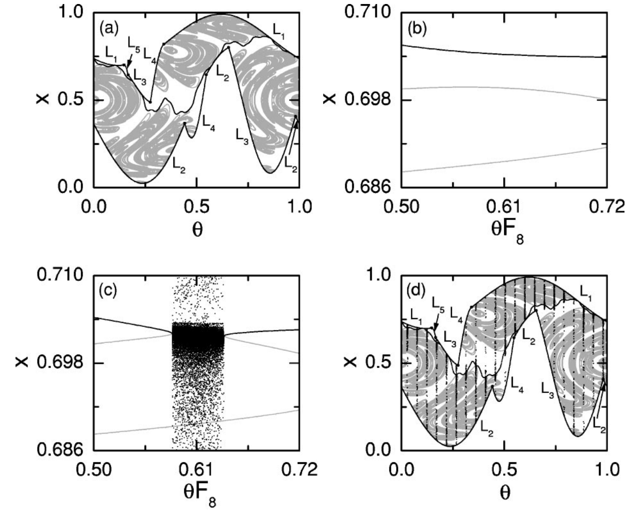

FIG. 3. (a) Smooth torus and ring-shaped unstable set in the RA of level 8 ($F_8=21$) for $a=3.38$ and $\varepsilon=0.586$. The ring-shaped unstable set (shown in gray) lies very close to the smooth torus (denoted by a black line) inside the absorbing area with boundary formed by segments of the critical curves L_k ($k=1,\ldots,5$) (the dots indicate where these segments connect). A magnified view near $(\theta F_8,x)=(0.61,0.698)$ is given in (b). (c) and (d). The eighth RA to the intermittent SNA for $a=3.38$ and $\varepsilon=0.5864$. The RA to the SNA is composed of the union of the periodic component and the intermittent chaotic component, where the latter occupies the 21 gaps in θ and is vertically bounded by portions of the critical curves L_k ($k=1,\ldots,5$) [a magnified gap near $\theta F_8=0.61$ is shown in (c)]. For more details, see the text.

curves L_k ($k=1,\ldots,4$). Note also that a ring-shaped unstable set, born via a phase-dependent saddle-node bifurcation and composed of eight small rings, lies inside the absorbing area. At first, each ring consists of the stable (shown in black) and unstable (shown in gray) orbits with forcing period F_6 ($=8$) [see the inset in Fig. 2(a)]. However, as the parameters increase such rings evolve, as shown in Fig. 2(b) for $a=3.26$ and $\varepsilon=0.46$. For fixed values of a and ε, the phase θ may be regarded as a "bifurcation parameter." As θ changes, a chaotic attractor appears through an infinite sequence of period-doubling bifurcations of stable periodic orbits in each ring, and then it disappears through a collision with the unstable F_6-periodic orbit [see the inset in Fig. 2(b)]. Thus, the attracting part (shown in black) of each ring consists of the union of the originally stable F_6-periodic attractor and the higher 2^nF_6-periodic ($n=1,2,\ldots$) and chaotic attractors born through the period-doubling process. On the other hand, the unstable part (shown in gray) of each ring is composed of the union of the originally unstable F_6-periodic orbit [e.g., the lower gray line in the inset in Fig. 2(b)] born via a saddle-node bifurcation and the destabilized F_6-periodic orbit [e.g., the upper gray line in the inset in Fig. 2(b)] born through a period doubling bifurcation. (As will be seen below, only the originally unstable F_6-periodic orbit may interact with the stable F_6-periodic orbit in the RA to the smooth torus through a saddle-node bifurcation.) With further increase in the parameters, both the size and the shape of the rings change, and for sufficiently large parameters, each ring consists of a large unstable part (shown in gray) and a small attracting part (shown in black), as shown in Fig. 2(c) for $a=3.326$ and $\varepsilon=0.526$. For the same parameter values as in Fig. 2(c), we increase the level of the RA to $k=8$. Then the number of rings ($=336$) increases significantly, and the unstable part [shown in gray and consisting of unstable orbits with period F_8 ($=21$)] becomes dominant because the attached attracting part (shown in black) becomes negligibly small [see Fig. 2(d)]. In this way, as the level k increases, the ring-shaped unstable set consists of a larger number of rings with a smaller attracting part (i.e., as the level k is increased, the unstable part of each ring becomes more and more dominant). Hence, it is conjectured that, in the quasiperiodic limit, these ring-shaped unstable sets might form a complicated invariant unstable set composed of only unstable orbits.

We now use RAs to explain the mechanism for the intermittent transition occurring in Figs. 1(b) and 1(c) for $a=3.38$. Figures 3(a) and 3(b) show that, inside the absorbing area, the ring-shaped unstable set (shown in gray) lies very close to the smooth torus (shown in black) for $\varepsilon=0.586$ in the RA of level $k=8$. As ε passes a threshold value ε_8 ($=0.586\,366$), a phase-dependent saddle-node bifurcation occurs through the collision of the smooth torus and the ring-shaped unstable set. As a result "gaps," where no orbits with period F_8 ($=21$) exist, are formed. A magnified gap is shown in Fig. 3(c) for $\varepsilon=0.5864$. Note that this gap is filled by intermittent chaotic attractors together with orbits with periods higher than F_8 embedded in very small windows. As shown in Fig. 3(d), the RA to the whole attractor consists of the union of the periodic component and the intermittent chaotic component, where the latter occupies the 21 gaps in θ and is vertically bounded by segments of the critical curves L_k $(k=1,\ldots,5)$. However, the periodic component dominates: the average Lyapunov exponent ($\langle\sigma_x\rangle=-0.09$) is negative, where $\langle\cdots\rangle$ denotes the average over the whole θ. We note that Fig. 3(d) resembles Fig. 1(c), although the level $k=8$ is low. Increasing the level to $k=15$, we find that the threshold value ε_k at which the phase-dependent saddle-node bifurcation occurs converges to the quasiperiodic limit ε^* $(=0.584\,726\,781)$ in an algebraic manner, $|\Delta\varepsilon_k|\sim F_k^{-\alpha}$, where $\Delta\varepsilon_k=\varepsilon_k-\varepsilon^*$ and $\alpha\simeq 2.2$. In the quasiperiodic limit $k\to\infty$, the RA to the attractor has a dense set of gaps that are filled by intermittent chaotic attractors and bounded by portions of the critical curves. Thus, an intermittent SNA, containing the ring-shaped unstable set and filling the absorbing area, appears, as shown in Fig. 1(c).

In addition to the transition to an intermittent SNA, we also find that as ε passes another threshold value ε_c $(=0.5848)$, the SNA transforms into a chaotic attractor with a positive Lyapunov exponent. Using the RA, this transition to chaos may be explained. For each RA to the attractor, its angle averaged Lyapunov exponent $\langle\sigma_x\rangle$ is given by the sum of the ''weighted'' Lyapunov exponents of its periodic and chaotic components, Λ_p and Λ_c, (i.e., $\langle\sigma_x\rangle=\Lambda_p+\Lambda_c$), where $\Lambda_{p(c)}\equiv M_{p(c)}\langle\sigma_x\rangle_{p(c)}$, and $M_{p(c)}$ and $\langle\sigma_{p(c)}\rangle$ are the Lebesgue measure in θ and average Lyapunov exponent of the periodic (chaotic) component, respectively. After passing a threshold value where the magnitude of Λ_p and Λ_c are balanced, the chaotic component becomes dominant, and hence a chaotic attractor appears.

III. SUMMARY

Using the RAs we have investigated the mechanism for the intermittent route to SNAs in the quasiperiodically forced logistic map. It has been found that when a smooth torus makes a collision with a new type of ring-shaped unstable set, a transition to an intermittent SNA occurs via a phase-dependent saddle-node bifurcation. In a future paper [20], it will be shown that other typical transitions such as interior [routes *b* and *c* in Fig. 1(a)] and boundary [routes *d* and *e* in Fig. 1(a)] crises may also occur near the main tongue through an interaction with the ring-shaped unstable set [20]. Furthermore, as ε decreases toward zero, similar tongues appear successively near the terminal points of the higher-order torus-doubling bifurcations, and band-merging crises may also occur through a collision with the ring-shaped unstable set [20]. Consequently, ring-shaped unstable sets play a central role for dynamical transitions occurring near the tongues. Finally, we note that these kinds of dynamical transitions seem to be ''universal,'' in that we observe that they occur in typical quasiperiodically forced period-doubling systems of different nature, such as the quasiperiodically forced Hénon map, ring map, and pendulum [20,21].

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