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Parameter-Mismatching Effect on the Attractor Bubbling in Coupled Chaotic Systems

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We investigate the effect of parameter mismatch on weak synchronization in unidirectionally coupled invertible Hénon maps. Due to the existence of positive local transverse Lyapunov exponents, a weakly stable synchronized attractor (SCA) becomes sensitive with respect to the variation of the mismatching parameter. As in coupled noninvertible one-dimensional maps, a quantifier, called the parameter sensitivity exponent (PSE), that measures the "degree" of such parameter sensitivity, is introduced. In terms of these PSEs, we characterize the parameter-mismatching effect on the attractor bubbling occurring in the regime of weak synchronization.

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Recently, because of its potential practical applications (*e.g.*, see [1]), the phenomenon of synchronization in coupled chaotic systems has become a field of intensive research. When identical chaotic systems synchronize, a synchronous chaotic attractor (SCA) exists on an invariant subspace of the whole phase space [2]. If the SCA is stable against a perturbation transverse to the invariant subspace, it may become an attractor in the whole phase space. Such transverse stability of the SCA is intimately associated with transverse bifurcations of periodic saddles embedded in the SCA [3–5]. If all periodic saddles are transversely stable, the SCA becomes asymptotically stable, and then we have "strong" synchronization. However, as the coupling parameter passes through a threshold value, a periodic saddle first becomes transversely unstable through a local bifurcation. Then, trajectories may be locally repelled from the invariant subspace when they visit the neighborhood of the transversely unstable periodic repeller. Thus, we have "weak" synchronization. For this case, transient intermittent bursting or basin riddling may occur depending on the global dynamics [4, 5]. Here, we are interested in the bursting case.

In a real situation, a small mismatch between the subsystems that destroys the invariant subspace is unavoidable. For the bursting case, any small mismatching results in a continual sequence of intermittent bursts, where the long period of the nearly synchronous state (laminar phase) is randomly interrupted by the short-time burst (burst phase). This attractor bubbling demonstrates the sensitivity of the weakly stable SCA with respect to the variation of the mismatching parameter. Recently, to characterize such parameter sensitivity, we have introduced a quantifier, called the parameter sensitivity exponent (PSE) measuring the "degree" of the parameter sensitivity in coupled noninvertible one-dimensional (1D) maps [6]. Here, we extend the method of characterizing the parameter sensitivity in terms of PSEs to high-dimensional invertible systems, and characterize the effect of parameter mismatch on the attractor bubbling.

As a representative model for the Poincaré maps of coupled chaotic oscillators, we consider unidirectionally coupled invertible Hénon maps:

$$T : \begin{cases} \mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \mathbf{y}_n) = \mathbf{f}(\mathbf{x}_n, a), \\ \mathbf{y}_{n+1} = \mathbf{G}(\mathbf{x}_n, \mathbf{y}_n) = \mathbf{f}(\mathbf{y}_n, b) + c\,\mathbf{g}(\mathbf{y}_n, \mathbf{x}_n), \end{cases} \quad (1)$$

where $\mathbf{x}_n$ [$= (x_n^{(1)}, x_n^{(2)})$] and $\mathbf{y}_n$ [$= (y_n^{(1)}, y_n^{(2)})$] are state variables of the subsystems at a discrete time n, local dynamics in each subsystem with a control parameter p ($p = a, b$) is governed by the Hénon map

$$\mathbf{f}(\mathbf{x}, p) = (f(x^{(1)}, a) - x^{(2)}, \beta\, x^{(1)});\ \ f(x, a) = 1 - a\, x^2, \quad (2)$$

c is a coupling parameter between the two subsystems, and $\mathbf{g}(\mathbf{x}, \mathbf{y})$ is a coupling function of the form

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) = (g(x^{(1)}, y^{(1)}), 0);\ \ g(x, y) = y^2 - x^2. \quad (3)$$

For this unidirectionally coupled system with a constant Jacobian determinant β^2 ($|\beta| < 1$), the first, master Hénon map with state variables $\mathbf{x}$ can be regarded as a driver for the second, slave or response Hénon map with state variables $\mathbf{y}$ through the coupling term. Here, we fix the value of β at $\beta = 0.1$.

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For the case of identical Hénon maps (*i.e.*, $a = b$), there is an invariant synchronization plane, $x^{(1)} = y^{(1)}$ and $x^{(2)} = y^{(2)}$, in the $x^{(1)}-x^{(2)}-y^{(1)}-y^{(2)}$ phase space. However, in the presence of a mismatching between the two Hénon maps, the synchronization plane is no longer invariant. To take into consideration such a mismatching effect, we introduce a small mismatching parameter ε in the coupled Hénon maps of Eq. (1), such that

$$b = a - \varepsilon, \tag{4}$$

and consider an orbit $\{(\mathbf{x}_n, \mathbf{y}_n)\}$ starting from an initial point on the synchronization plane (*i.e.*, $\mathbf{x}_0 = \mathbf{y}_0$). As the strength of the local transverse repulsion from the synchronization plane increases, the SCA becomes more and more sensitive with respect to the variation of ε. Such parameter sensitivity of the SCA for $\varepsilon = 0$ may be characterized by calculating the derivative of the transverse variable $\mathbf{u}_n$ $(= \mathbf{x}_n - \mathbf{y}_n)$, denoting the deviation from the synchronization plane, with respect to ε (*i.e.* $\frac{\partial \mathbf{u}_{n+1}}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial \mathbf{x}_{n+1}}{\partial \varepsilon}\Big|_{\varepsilon=0} - \frac{\partial \mathbf{y}_{n+1}}{\partial \varepsilon}\Big|_{\varepsilon=0}$). By using Eq. (1), we may obtain a recurrence relation

$$\frac{\partial \mathbf{u}_{n+1}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = r(\mathbf{x}_n^*)\, \frac{\partial \mathbf{u}_n}{\partial \varepsilon}\bigg|_{\varepsilon=0} + \mathbf{f}_a(\mathbf{x}_n^*, a), \tag{5}$$

where $\frac{\partial \mathbf{u}_n}{\partial \varepsilon}\Big|_{\varepsilon=0} = \left(\frac{\partial u_n^{(1)}}{\partial \varepsilon}\Big|_{\varepsilon=0}, \frac{\partial u_n^{(2)}}{\partial \varepsilon}\Big|_{\varepsilon=0} \right)$, and the 2×2 matrix $r(\mathbf{x}_n^*)$ is given by

$$r(\mathbf{x}_n^*) \equiv \begin{pmatrix} f_{x^{(1)}}(x_n^{(1)*}, a) - c\, h(x_n^{(1)*}) & -1 \\ \beta & 0 \end{pmatrix} \tag{6}$$

and

$$\mathbf{f}_a(\mathbf{x}_n^*, a) = \begin{pmatrix} f_a(x_n^{(1)*}, a) \\ 0 \end{pmatrix}. \tag{7}$$

Here, f_x and f_a are the derivatives of $f(x, a)$ with respect to x and a, $\{(\mathbf{x}_n^*, \mathbf{y}_n^*)\}$ is a synchronous orbit with $\mathbf{x}_n^* = \mathbf{y}_n^*$ for $\varepsilon = 0$, and $h(x)$ is a reduced coupling function defined by [7]

$$h(x) \equiv \frac{\partial g(x, y)}{\partial y}\bigg|_{y=x}. \tag{8}$$

Hence, starting from an initial orbit point $(\mathbf{x}_0^*, \mathbf{y}_0^*)$ on the synchronization plane, we may obtain derivatives at all points of the orbit:

$$\frac{\partial \mathbf{u}_N}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \sum_{k=1}^{N} R_{N-k}(\mathbf{x}_k^*) \mathbf{f}_a(\mathbf{x}_{k-1}^*, a) + R_N(\mathbf{x}_0^*)\, \frac{\partial \mathbf{u}_0}{\partial \varepsilon}\bigg|_{\varepsilon=0}, \tag{9}$$

where $R_M(\mathbf{x}_m^*) = \prod_{i=0}^{M-1} r(\mathbf{x}_{m+i}^*)$ is a product of the "transverse Jacobian matrices" $r(\mathbf{x})$ determining the stability against a perturbation transverse to the synchronization plane and $R_0 = I$ (identity matrix). One can easily show that the eigenvalues, $\lambda_M^{T,1}(\mathbf{x}_m^*)$ and $\lambda_M^{T,2}(\mathbf{x}_m^*)$ $(|\lambda_M^{T,1}(\mathbf{x}_m^*)| > |\lambda_M^{T,2}(\mathbf{x}_m^*)|)$, of $R_M(\mathbf{x}_m^*)$ are associated with local (M-time) transverse Lyapunov exponents $\sigma_M^{T,1}$ and $\sigma_M^{T,2}$ $(\sigma_M^{T,1} > \sigma_M^{T,2})$ of the SCA that are averaged over M synchronous orbit points starting from $\mathbf{x}_m^*$ as follows:

$$\sigma_M^{T,i}(\mathbf{x}_m^*) = \frac{1}{M} \ln |\lambda_M^{T,i}(\mathbf{x}_m^*)|, (i = 1, 2). \tag{10}$$

Thus, $\lambda_M^{T,1}$ and $\lambda_M^{T,2}$ become local (transverse stability) multipliers that determine local sensitivity of the motion during a finite time M. As $M \to \infty$, $\sigma_M^{T,1}$ approaches the largest transverse Lyapunov exponent σ_T^1 that denotes the average exponential rate of divergence of an infinitesimal perturbation transverse to the SCA. Eq. (9) reduces to

$$\frac{\partial \mathbf{u}_N}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \mathbf{S}_N(\mathbf{x}_0^*) \equiv \sum_{k=1}^{N} R_{N-k}(\mathbf{x}_k^*) \mathbf{f}_a(\mathbf{x}_{k-1}^*, a), \tag{11}$$

because $\frac{\partial \mathbf{u}_0}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0$. In the case of weak synchronization, there are transversely unstable periodic repellers embedded in the SCA. When a typical trajectory visits neighborhoods of such repellers, it has segments experiencing local repulsion from the synchronization plane. Thus, the distribution of largest local transverse Lyapunov exponents $\sigma_M^{T,1}$ for a large ensemble of trajectories and large M may have a positive tail. For the segments of a trajectory exhibiting a positive largest local Lyapunov exponent $(\sigma_M^{T,1} > 0)$, the largest local transverse multipliers $\lambda_M^{T,1}$ $[= \pm \exp(\sigma_M^{T,1} M)]$ can be arbitrarily large, and hence the partial sums $S_N^{(i)}$ $(i = 1, 2)$ may be arbitrarily large. This implies unbounded growth of the derivatives $\frac{\partial u_N^{(i)}}{\partial \varepsilon}\Big|_{\varepsilon=0}$ $(i = 1, 2)$ as N tends to infinity, and consequently the weakly stable SCA may have a parameter sensitivity.

As an example, we consider the SCA for $a = 1.8$ in unidirectionally coupled Hénon maps. A strongly stable SCA exists in the interval of $c_{t,l}$ $(= -2.9) < c < c_{t,r}$ $(= -0.7)$. For this case of strong synchronization, there is no parameter sensitivity, because all periodic saddles embedded in the SCA are transversely stable. However, when the coupling parameter c passes a threshold value $c_{t,r}$, a saddle fixed point first becomes transversely unstable via a period-doubling bifurcation, and then the SCA becomes weakly stable. For this case, the weakly stable SCA has a parameter sensitivity, because of local transverse repulsion of the periodic repellers embedded in the SCA. Thus, however small the parameter mismatching ε, a persistent intermittent bursting, called attractor bubbling, occurs, as shown in Figs. 1(a) and 1(b) for $c = -0.62$ and $\varepsilon = 0.001$. As c is varied away from $c_{t,r}$, transversely unstable periodic repellers appear successively in the SCA through transverse bifurcations. Then, the degree of the parameter sensitivity of the SCA increases, because of the increase in the strength of local transverse repulsion of the periodic repellers.

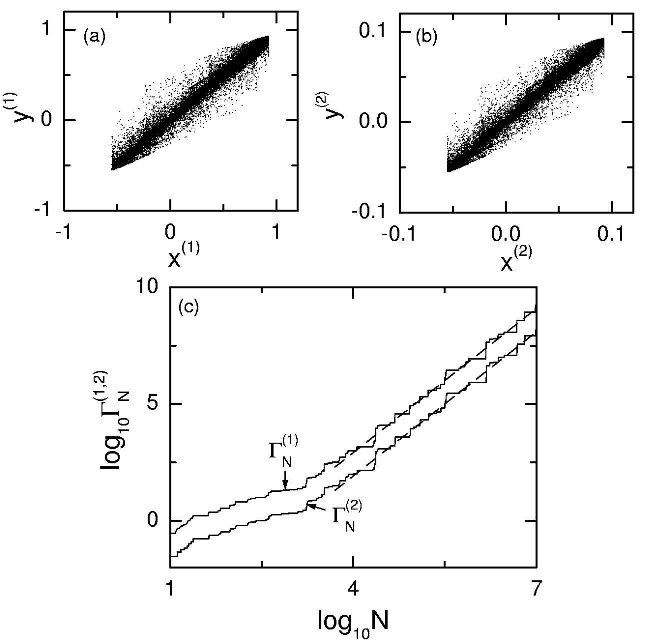

Fig. 1. Effect of parameter mismatch with $\varepsilon = 0.001$ on the chaos synchronization for $a = 1.8$ and $c = -0.62$. Projections of a bubbling attractor onto (a) the $x^{(1)} - y^{(1)}$ plane, and (b) the $x^{(2)} - y^{(2)}$ plane. (c) Log-log plot of parameter sensitivity functions $\Gamma_N^{(1)}$ and $\Gamma_N^{(2)}$ for $a = 1.8$ and $c = -0.62$. Each $\Gamma_N^{(i)}$ $(i = 1, 2)$ is well fitted with a dashed line with slope 2.15.

To quantitatively characterize the parameter sensitivity of the SCA, we iterate Eqs. (1) and (5) starting from an initial orbit point $(\mathbf{x}_0^*, \mathbf{y}_0^*)$ on the synchronization plane and $\frac{\partial \mathbf{u}_0}{\partial \varepsilon}\big|_{\varepsilon=0} = 0$, and then we obtain the partial sum $\mathbf{S}_N(\mathbf{x}_0^*)$ of Eq. (11). The quantity $\mathbf{S}_N$ $[= \frac{\partial \mathbf{u}_N}{\partial \varepsilon}\big|_{\varepsilon=0}]$ becomes very intermittent. However, on looking only at the maximum

$$\gamma_N^{(i)}(x_0^*) = \max_{0 \le n \le N} |S_n^{(i)}(\mathbf{x}_0^*)| \ \ (i = 1, 2), \tag{12}$$

one can easily see the boundedness of $S_N^{(i)}$. For this case $\gamma_N^{(1)}$ and $\gamma_N^{(2)}$ grow unboundedly, and hence the weakly stable SCA has a parameter sensitivity. The growth rate of the function $\gamma_N^{(i)}(\mathbf{x}_0^*)$ with time N represents the degree of the parameter sensitivity, and can be used as a quantitative characteristic of the weakly stable SCA. However, $\gamma_N^{(i)}(\mathbf{x}_0^*)$ depends on a particular trajectory. To obtain a representative quantity, we consider an ensemble of randomly chosen initial points $(\mathbf{x}_0^*, \mathbf{y}_0^*)$ on the synchronization plane, and take the minimum value of $\gamma_N^{(i)}$ with respect to the initial orbit points,

$$\Gamma_N^{(i)} = \min_{\mathbf{x}_0^*} \gamma_N^{(i)}(\mathbf{x}_0^*) \ \ (i = 1, 2). \tag{13}$$

Figure 1(c) shows parameter sensitivity functions $\Gamma_N^{(1)}$ and $\Gamma_N^{(2)}$ for $c = -0.62$. Note that $\Gamma_N^{(1)}$ and $\Gamma_N^{(2)}$ grow unboundedly with the same power δ,

$$\Gamma_N^{(i)} \sim N^{\delta} \text{ for } i = 1, 2, \tag{14}$$

because their growth is governed by the same largest local multipliers $\lambda_M^{T,1}$. Here the value $\delta \simeq 2.15$ is a quantitative characteristic of the parameter sensitivity of the SCA, and we call this the PSE. By increasing the coupling parameter from the bubbling transition point $c_{t,r}$, we obtain the PSEs. To obtain satisfactory statistics, we consider 100 ensembles for each c, each of which contains 100 randomly chosen initial orbit points, and choose the average value of the 100 PSEs obtained in the 100 ensembles. Figure 2(a) shows the plot of such PSEs versus c. Note that the PSE δ monotonically increases as c is varied away from $c_{t,r}$, and tends to infinity as c approaches the blow-out bifurcation point c_b $(\simeq -0.6026)$. This increase in the parameter sensitivity of the SCA is caused by the increase in the strength of local transverse repulsion of the periodic repellers embedded in the SCA. After the blow-out bifurcation, the weakly stable SCA becomes transversely unstable, and hence a complete desynchronization occurs.

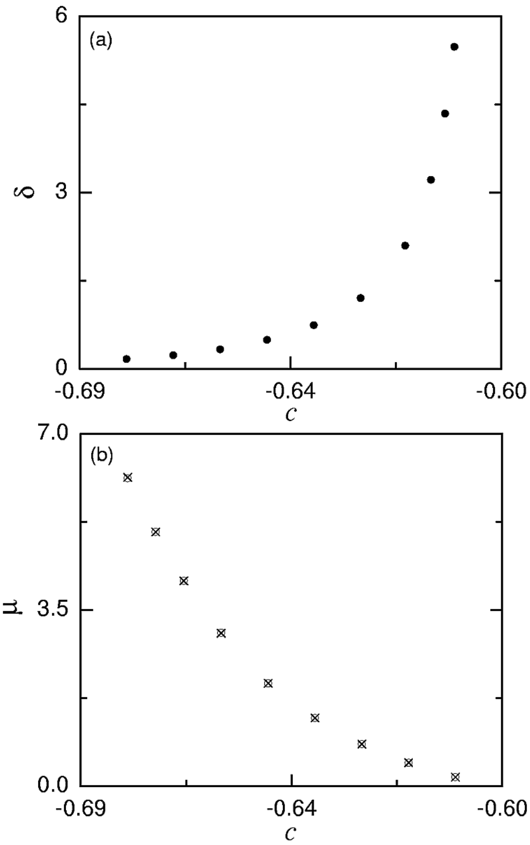

Fig. 2. (a) Plot of the PSEs δ (solid circles) versus c for $a = 1.8$. (b) Plot of the laminar phase exponents (LPEs) μ (open circles) versus c for $a = 1.8$. They agree well with the reciprocal of the PSEs (crosses).

We now characterize the parameter-mismatching effect on the attractor bubbling in terms of the PSEs. For the bubbling case, the quantity of interest is the average interburst time τ that a typical trajectory spends near the synchronization plane. As c is varied from the bubbling transition point, such average time becomes short because the strength of local transverse repulsion of the periodic repellers embedded in the SCA increases. For this case, the bubbling attractor is in the laminar phase when the magnitude of the deviation from the synchronization plane, d_n $[\equiv (|u_n^{(1)}| + |u_n^{(2)}|)/2]$, is less than a threshold value d^* (*i.e.*, $d_n < d^*$). Otherwise, it is in the bursting phase. Here, d^* is very small compared to the maximum bursting amplitude, and this is the maximum deviation from the synchronization plane that may be acceptable in the context of synchronization. For each c, we follow the trajectory starting from the initial condition $(0.2, 0.1\,x_0^{(1)}, 0.2, 0.1\,y_0^{(1)})$ until 50,000 laminar phases are obtained, and then we get the average laminar length τ (*i.e.*, the average interburst interval) that scales with ε as

$$\tau \sim \varepsilon^{-\mu}, \tag{15}$$

where μ refers to the laminar phase exponent (LPE). The plot of the LPE μ versus c is shown in Fig. 2(b). As c increases, the value of μ decreases, because the average laminar length shortens. Note that this LPE μ is associated with the PSE δ as follows. For a given ε, consider a trajectory starting from a randomly chosen initial orbit point on the synchronization plane. Then, from Eq. (14) the "average" time τ at which the magnitude of the deviation from the synchronization plane becomes the threshold value d^* can be obtained:

$$\tau \sim \varepsilon^{-1/\delta}. \tag{16}$$

Thus, the two exponents have a reciprocal relation,

$$\mu = 1/\delta. \tag{17}$$

The reciprocal values of δ are also plotted in Fig. 2(b), and they agree well with the values of μ.

In summary, we have introduced a quantifier, called the PSE, to quantitatively measure the degree of sensitivity of the SCA with respect to the variation of the mismatching parameter in coupled invertible Hénon maps. In terms of these PSEs, the parameter-mismatching effect on the attractor bubbling has been characterized. It has thus been found that the scaling exponent for the average interburst time is given by the reciprocal of the PSE. Besides the parameter sensitivity, the weakly stable SCA becomes sensitive with respect to noise. This noise sensitivity can also be characterized in terms of noise sensitivity exponents (NSEs), as in coupled 1D maps [8]. Thus, the method of characterizing the parameter and noise sensitivity of the weakly stable SCA in terms of PSEs and NSEs has been extended to high-dimensional invertible systems such as coupled Hénon maps and pendula [9].

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