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Critical behavior in coupled nonlinear systems

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We study the critical behavior of period-doubling bifurcations in two coupled one-dimensional maps. In a linear-coupling case, in which the coupling function has a leading linear term, the set of critical points, called the critical set, consists of an infinite number of critical line segments and the zero coupling point, whereas only one critical line segment constitutes the critical set in the case of a nonlinear coupling whose leading term is nonlinear. We find three (two) kinds of critical behaviors in the linear-(nonlinear-) coupling case, depending on the position on the critical set.

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Since the period-doubling route to chaos in one-dimensional (1D) dissipative [1] and two-dimensional area-preserving [2] maps was known, efforts have been made in studies of coupled maps to attempt to generalize to higher-dimensional dissipative [3–6] and volume-preserving [7,8] maps. Here we are interested in the transition to chaos via an infinite sequence of period-doubling bifurcations (PDB) in coupled 1D (dissipative) maps. In two coupled [3] and many coupled [4–6] maps, two additional scaling factors, associated with the coupling-strength parameter, have been found at the (critical) zero coupling point, in addition to the parameter scaling factor δ of 1D maps [1].

In this paper, using the scaling-matrix method [7,8], we study the critical behavior of PDB in two coupled 1D maps. Unlike the previous work [3], we find an infinite number of critical points. In a linear-coupling case, an infinite set of critical line segments, together with the previously found zero coupling point [3], constitute the critical set, whereas in a nonlinear-coupling case, the critical set consists of the only one critical line segment, one end of which is the zero coupling point. The critical behavior depends on the position on the critical set. We find two kinds of new critical behaviors at each critical line segment in the linear-coupling case and one kind of new critical behavior at interior points of the critical line in the nonlinear-coupling case, in addition to the critical behavior at the zero coupling point.

We consider a map T consisting of two identical 1D maps coupled symmetrically:

$$T: \begin{cases} x_{i+1} = F(x_i, y_i) \equiv f(x_i) + g(x_i, y_i) \\ y_{i+1} = F(y_i, x_i) \equiv f(y_i) + g(y_i, x_i) , \end{cases} \quad (1)$$

where the subscript i designates the discrete time, $f(x)$ is the uncoupled 1D map with a quadratic extremum, and $g(x,y)$ is the coupling function, obeying the condition $g(x,x)=0$. Various periodic orbits exist in the coupled system and can be classified in terms of the phase difference between the motions in x and y space: in-phase and out-of-phase orbits [3]. In this paper we study only in-phase orbits. An in-phase orbit satisfies $x_i = y_i$ for all i's, and hence, it can easily be found from the uncoupled 1D map equation, since $g(x,x)=0$ for all x.

The stability of an in-phase orbit of period p is determined from the p product of the linearized map DT along the orbit:

$$DT = \begin{pmatrix} f'(x) - G(x) & G(x) \\ G(x) & f'(x) - G(x) \end{pmatrix} ,$$

where the prime denotes the derivative and $G(x) = \partial g(x,y)/\partial y|_{y=x}$. Then the pair of stability multipliers of the orbit is

$$\lambda_1 = \prod_{i=1}^{p} f'(x_i), \quad \lambda_2 = \prod_{i=1}^{p} [f'(x_i) - 2G(x_i)] . \quad (2)$$

Notice that λ_1 is merely the same as the 1D case and the coupling affects only λ_2.

An in-phase orbit loses the stability by either PDB or tangent bifurcation. Between them, the successive PDB complete an infinite sequence. Our concern here is on this period-doubling route to chaos and its critical behavior at the transition. We choose $f(x) = 1 - Ax^2$ as the uncoupled map and consider two kinds of couplings, linear- and nonlinear-coupling cases; a coupling is called linear or nonlinear according to its leading term.

As an example of the linear-coupling case we consider linearly coupled maps, i.e., $g(x,y) = c(y-x)$. Figure 1 shows the stability diagram of in-phase orbits. Each in-phase orbit of level n (period 2^n) loses its stability at the horizontal boundary line of its stable region via PDB, giving rise to the creation of a period doubled in-phase orbit of level $n+1$. Such an infinite sequence is terminated at a finite value of A_∞ ($=1.401\,155\ldots$), the accumulation point of the 1D map.

The treelike structure of the stability diagram is an infinite pile of U-shape areas and rectangular shape areas. The U-shape branching is repeated at one side of each U-shape area, while changing the branching side alternatively. This branching side contains the zero coupling point, and it will be referred to as the zero c side. The other side of each U-shape area grows like a chimney without any further branchings (as an example, see the leftmost

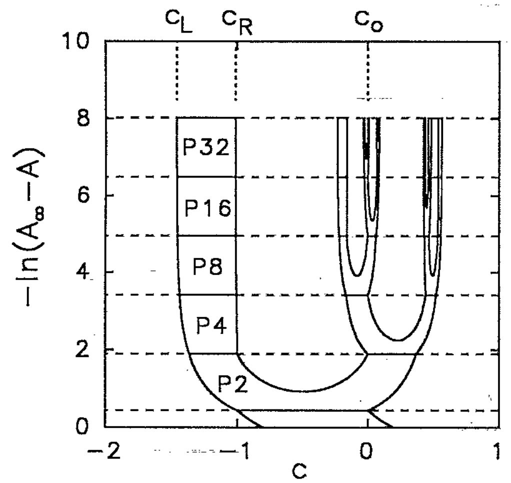

FIG. 1. Stability diagram of in-phase orbits for the map with the coupling $g(x,y)=c(y-x)$. PN designates the stable region of the in-phase orbit of period N, respectively. Each level of horizontal lines accumulating to A_∞ corresponds to the 1D map bifurcation point.

branch in Fig. 1). Note that this rule governs the asymptotic behavior of the tree structure even though there are a few exceptions for lower-level ($n \leq 8$) periodic orbits. There may appear other type of U-shape areas without the zero c side in the lower-level stability regions (e.g., the rightmost U-shape area in the third-level stability region). However, the U-shape branching for this kind of U-shape area terminates at some finite level and, after that, each side of the U-shape area grows like a chimney.

A sequence of (connected) stability regions with increasing period is called a bifurcation route [7,8]. There are two kinds of bifurcation routes. The sequence of the U-shape areas converges to the zero coupling point ($c=0$) on the $A=A_\infty$ line. It will be referred to as the U route. On the other hand, the sequence of rectangular areas in each chimney converges to a critical line segment on the $A=A_\infty$ line; for example, the leftmost one is denoted as $\overline{c_L c_R}$ joining two points, $c_L(=-1.457727\ldots)$ and $c_R(=-1.013402\ldots)$ on the $A=A_\infty$ line. This kind of route will be called a C route. Note that there are infinitely many C routes, while the U route is unique. An infinite set of critical line segments, therefore, together with the zero coupling point denoted as c_0 in the figure, constitute the critical set.

To study critical behavior, first choose a bifurcation route. Then a (self-similar) bifurcation path within the chosen route is formed by following the parameters (A_n,c_n) at which the orbit of level n (period 2^n) has some given stability multiplier values (λ_1,λ_2) [7,8]. The scaling behavior of the sequence $[(A_n,c_n), n=0,1,2,\ldots]$ can be determined by the scaling matrix method [7,8]. The 2×2 scaling matrix of level n, Γ_n, is defined as follows:

$$\begin{pmatrix} A_{n-1}-A_{n-2} \\ c_{n-1}-c_{n-2} \end{pmatrix} = \Gamma_n \begin{pmatrix} A_n - A_{n-1} \\ c_n - c_{n-1} \end{pmatrix}.$$

As $n\to\infty$ the eigenvalues of Γ_n converge to the limits, γ_1 and γ_2, which are just the parameter scaling factors of the sequence. With these scaling factors the sequence converges to an accumulation point (A_∞,c_∞). At this critical point, the stability multipliers, $\lambda_{1,n}$ and $\lambda_{2,n}$, of the orbit of level n converge to the critical stability multipliers, λ_1^* and λ_2^*. Since λ_1 depends only on A, λ_1^* is always the same as the 1D map value, i.e., $\lambda_1^*=-1.601191\ldots$ [1].

We follow the periodic orbits up to level 13, and the convergence of the eigenvalues of Γ_n to the scaling factors is shown in Tables I and II. We find three kinds of scaling behaviors dependent on the position on the critical set: c_0, two ends of each line segment (e.g., c_L,c_R), and the interior points of each line segment.

All bifurcation paths in the U route converge to the zero coupling point, (A_∞,c_0). The parameter scaling factors at the zero coupling point are just the 1D map constants (see Table I), i.e., $\gamma_1=\delta$ $(=4.669201\ldots)$ and $\gamma_2=\alpha$ $(=-2.502907\ldots)$. Since the coupling is zero at (A_∞,c_0), $\lambda_{2,n}$ converges to that of 1D map in the same way as $\lambda_{1,n}$ does, and thus $\lambda_2^*=\lambda_1^*$ (see Table I). This critical behavior at the zero coupling point has also been found by the renormalization method [3].

Two kinds of new critical behaviors are found at each

TABLE I. We followed, in the U route, a sequence of parameters (A_n,c_n), at which the pair of stability multipliers, $(\lambda_{1,n},\lambda_{2,n})$, of the orbit of level n (period 2^n) is $(-1,0)$. This sequence converges to the zero coupling point, (A_∞,c_0), with the scaling factors shown in the second and third columns. At the zero coupling point, a convergence of the second multiplier to its critical value, λ_2^*, is shown in the fourth column.

n	γ_1	γ_2	λ_2
6	4.6653920	−2.5024660	−1.60119124
7	4.6684036	−2.5028145	−1.60119134
8	4.6690284	−2.5028876	−1.60119133
9	4.6691648	−2.5029035	−1.60119133
10	4.6691937	−2.5029069	−1.60119133
11	4.6691999	−2.5029077	−1.60119133
12	4.6692012	−2.5029078	−1.60119133
13	4.6692015	−2.5029079	−1.60119133

TABLE II. We followed, in the leftmost C route, a sequence of parameters (A_n,c_n), at which the stability multipliers, $\lambda_{1,n}$ and $\lambda_{2,n}$, of the orbit of level n (period 2^n) are -1 and 0.9, respectively. This sequence converges to the left end point of the critical line segment, (A_∞,c_L), with the scaling factors shown in the second and third columns. At the critical point, convergence of the second multiplier to its critical value, λ_2^*, is shown in the fourth column.

n	γ_1	γ_2	λ_2
6	4.6609558	1.9476684	0.99997901
7	4.6674591	2.0022035	1.00000457
8	4.6688222	1.9972482	0.99999900
9	4.6691211	1.9998223	1.00000022
10	4.6691843	1.9997148	0.99999995
11	4.6691979	1.9999131	1.00000001
12	4.6692008	1.9999471	1.00000000
13	4.6692014	1.9999762	1.00000000

critical line segment. In each C route there exist two kinds of bifurcation paths: the one converging to the left end point of the critical line segment, e.g., (A_∞, c_L), and the other converging to the right end point, e.g., (A_∞, c_R). At both ends of each critical line segment, the parameter scaling factors become $\gamma_1 = \delta$ and $\gamma_2 = 2$ (see Table II). Note that the value of γ_2 is different from that ($\gamma_2 = \alpha$) at the zero coupling point. At these critical points λ_2^* is one (see Table II).

For any fixed value of c inside each critical line segment, $\lambda_{2,n}$ converges to zero. That is, all the interior points of the line segments are critical points with $\lambda_2^* = 0$. Figure 2 shows the behavior of $\lambda_{2,n}$ near the critical line segment, $\overline{c_L c_R}$. As shown in the figure, λ_2^* changes abruptly from zero to one at both ends. Since $\lambda_2^* = 0$ at interior points, critical behavior inside the critical line segment is the same as that of the 1D map, like the 2D Hénon-map case [9]. Therefore there exists only one scaling factor δ, associated with the nonlinearity strength parameter A.

The nonlinear-coupling case is studied with an example of $g(x,y) = c(y^2 - x^2)$. The stability diagram is shown in Fig. 3. Unlike the linear-coupling case, the critical set consists of the only one critical line segment $\overline{c_0 c_0'}$ joining two points, c_0 and c_0' $(= -A_\infty)$, on the $A = A_\infty$ line. As in the C routes, there exist two kinds of bifurcation paths: the one converging to the left end point, (A_∞, c_0'), and the other converging to the right end point, (A_∞, c_0).

At the zero coupling point, (A_∞, c_0), the parameter scaling factors are $\gamma_1 = \delta$ and $\gamma_2 = 2$ (the convergence of the eigenvalues of Γ_n to the scaling factors is shown in Table III), and $\lambda_2^* = \lambda_1^*$ since the critical map at (A_∞, c_0) becomes uncoupled. The value of γ_2 is, however, different from that in the linear-coupling case ($\gamma_2 = \alpha$), and hence, the critical behavior at the zero coupling point depends on the nature of coupling [3].

At the left end point, $(A_\infty, -A_\infty)$, we obtain the same scaling factors as those at the zero coupling point, because the second-iterate map becomes uncoupled at this critical point: $x_{t+2} = f(f(x_t))$, $y_{t+2} = f(f(y_t))$. From Eq. (2), in this quadratic-coupling case the stability multipliers of the orbit of period 2^n become

$$\lambda_{1,n} = \prod_{i=1}^{p} f'(x_i), \quad \lambda_{2,n} = \left(1 + \frac{2c}{A}\right)^p \lambda_{1,n}, \tag{3}$$

where $p = 2^n$. At the left end point $\lambda_{2,n}$ becomes $\lambda_{1,n}$ for $n \geq 1$, and hence, $\lambda_2^* = \lambda_1^*$. Therefore the critical behavior at both ends of the critical line are the same.

A new kind of critical behavior is found at interior points of the critical line. For any fixed value of c inside the critical line $(-A_\infty < c < 0)$, $\lambda_{2,n}$ in Eq. (3) converges to zero as $n \to \infty$, and consequently, all the interior points become critical points with $\lambda_2^* = 0$. Therefore the critical behavior at interior points is the same as that inside each critical line segment in the linear-coupling case.

We have also studied several other coupling cases with the coupling function, $g(x,y) = c(y-x)(d+ex+fy)$. Here d, e, and f are arbitrary constants. In all the linear (d is nonzero) and nonlinear ($d=0$) cases studied, the critical behavior is found to be the same as the above

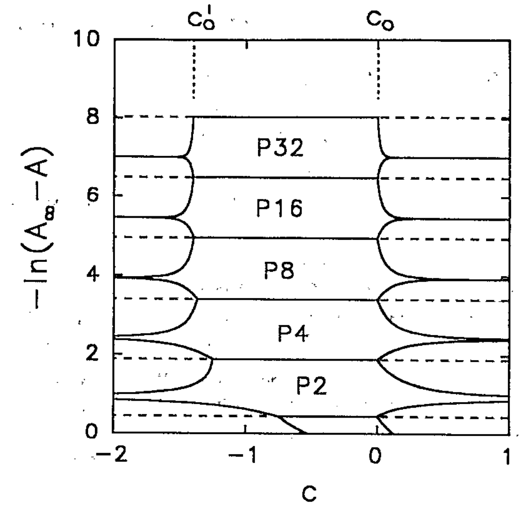

FIG. 3. Stability diagram of in-phase orbits for the map with the coupling $g(x,y) = c(y^2 - x^2)$. PN designates the stable region of the in-phase orbit of period N, respectively. Each level of horizontal lines accumulating to A_∞ corresponds to the 1D map bifurcation point.

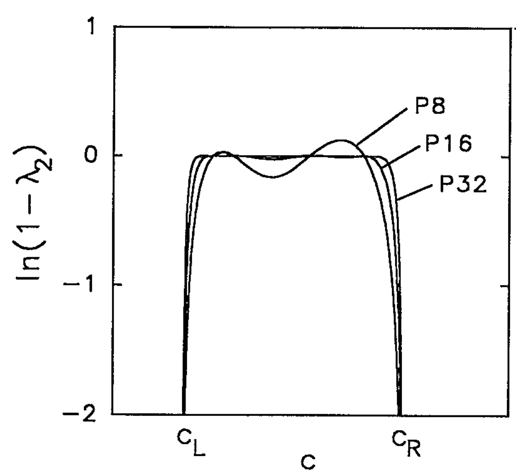

FIG. 2. Behavior of $\lambda_{2,n}(A_\infty, c_\infty)$ which converge to λ_2^* very rapidly as the period of the orbit becomes longer. PN designates the period of the orbit.

TABLE III. In the case, $g(x,y) = c(y^2 - x^2)$, we followed a sequence of parameters (A_n, c_n), at which the pair of stability multipliers, $(\lambda_{1,n}, \lambda_{2,n})$, of the orbit of level n (period 2^n) is $(-1, -0.8)$. This sequence converges to the zero coupling point with the scaling factors shown in the second and third columns.

n	γ_1	γ_2
6	4.6608087	2.0025182
7	4.6674604	1.9997950
8	4.6688219	1.9995814
9	4.6691211	1.9997228
10	4.6691843	1.9998469
11	4.6691979	1.9999204
12	4.6692008	1.9999595
13	4.6692014	1.9999796

linearly-coupled and quadratic-coupling cases, respectively.

These numerical results suggest that there exist three fixed maps of the renormalization transformation in the function space of coupled maps. It has been found that a fixed map, associated with the critical behavior at the zero coupling point, has three relevant eigenvalues, δ, α, and 2 [3]. We find two additional fixed maps, associated with the new critical behavior. The first one, associated with the critical behavior at both ends of each critical line segment in the linear coupling case, has two relevant eigenvalues, δ and 2, and the second one, which has only one relevant eigenvalue δ, is associated with the critical behavior at interior points of each critical line segment in the linear- and nonlinear-coupling cases. A detailed account of renormalization analysis will be given elsewhere [10].

In summary, using the scaling matrix method, we study the critical behavior of PDB in two coupled 1D maps. New critical behavior is found at each critical line segment in the linear-coupling case and at interior points of the critical line for the nonlinear-coupling case, as well as the previously known critical behavior at the zero coupling point.

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