Recurrence of invariant circles in a dissipative standardlike map

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We study a dissipative standardlike map that contains a parameter z that can be used to tune the map from a piecewise-linear map ($z=0$) to the dissipative standard map ($z=\infty$). When both the tuning parameter z and dissipation are small, the reappearance of an invariant circle after its breakup is observed. This recurrence phenomenon takes place as the nearby mode-locked resonances separate after they overlapped. However, as the dissipation is increased, the number of recurrences gradually decreases, and ultimately reappearance ceases at some dissipation parameter value dependent on z. Scaling behavior at the disappearance and reappearance points is also discussed.

I. INTRODUCTION

In recent years, much attention has been paid to the transition to chaos in dissipative dynamical systems with two competing frequencies [1–6]. A model for this transition is the dissipative standard map [4–6].

$$T:\begin{cases} r_{n+1}=br_n+kg(\theta_n) \\ \theta_{n+1}=\theta_n+\Omega+r_{n+1}\ , \end{cases} \tag{1.1}$$

where

$$g(\theta)=g_s(\theta)=-(1/2\pi)\sin(2\pi\theta)\ .$$

This map has a constant Jacobian b. For the conservative case ($b=1$), this is just the standard map; for the infinitely dissipative case ($b=0$), it reduces the sine-circle map. When the map is dissipative, there is a single quasiperiodic transition to chaos [3–6]. Universal properties of this transition have also been studied [3–6].

Wilbrink [7] studied the conservative case ($b=1$) of a two-parameter standardlike map where the function g_s is replaced by

$$g_w(\theta)=-\frac{\sqrt{1+z}}{2\pi}\arcsin\left[\frac{\sin(2\pi\theta)}{\sqrt{1+z}}\right]. \tag{1.2}$$

He found that for small z values, invariant circles can reappear after they have disappeared. This is in contrast to the standard map where there is no reappearance of invariant circles. Study of the quasiperiodic Schrödinger operator [8], the extended Frenkel-Kondorova mode [9], and the extended standard map [10] also gives similar results.

In this paper we study the dissipative case of the Wilbrink map where the function g_s in Eq. (1.1) is replaced by the function g_w in Eq. (1.2). By varying the parameter z, the function g_w is tuned from a piecewise-linear function [7] ($z=0$) to the function $g_s(z=\infty)$, and it is always analytic except when $z=0$. We study the analytic case $z>0$. In the area-preserving case ($b=1$), invariant circles can reappear; in the infinitely dissipative case ($b=0$), there is no recurrence of invariant circles. We will study the transition between the conservative and dissipative cases [11,12] by varying the dissipation parameter from $b=1$ to $b=0$.

In this dissipative Wilbrink map, we study the breakup of an invariant circle whose rotation number is the inverse golden mean. When both the tuning parameter z and dissipation are small, it is found that the invariant circle can reappear after it disappeared when the nonlinearity is increased. However, as the dissipation is increased, the number of recurrences gradually decreases, and ultimately recurrence ceases at some dissipation parameter value b^* dependent on z. Thus, when $b\le b^*$, there is no recurrence of invariant circles.

The recurrence of an invariant circle can be understood in terms of the separation of the nearby mode-locked resonances after they overlap. This is in contradistinction to the dissipative standard map case. This separation is crucial for the recurrence of an invariant circle since it can exist only if there is no overlap of the nearby resonances.

This paper is organized as follows: In Sec. II the method of finding out if an invariant circle exists will be discussed. In Sec. III we will study the breakup of invariant circles. In Sec. IV scaling behavior at the disappearance and reappearance points will be investigated. In Sec. V a summary is given.

II. METHODOLOGY

Since the map (1.1) with $g(\theta)=g_w(\theta)$ is periodic in θ with period 1, one can identify θ and $\theta+1$ and think of the map as acting on a cylinder. The rotation number of a trajectory starting at (r_0,θ_0) is defined by

$$\omega=\lim_{n\to\infty}(\theta_n-\theta_0)/n\ , \tag{2.1}$$

if the limit exists. Periodic orbits of period q have rational rotation numbers p/q if $\theta_q=\theta_0+p$. It is often useful to expand this rotation number as a continued fraction,

$$\omega=m_0+\cfrac{1}{m_1+\cfrac{1}{m_2+\cdots}}\equiv[m_0,m_1,m_2,\ldots] . \quad (2.2)$$

The breakup of an invariant circle with an irrational rotation number ω is most easily tracked numerically by studying the sequence of periodic orbits whose rotation numbers are the successive convergents of ω. Here we choose to study a particular rotation number, the reciprocal of the golden mean,

$$\zeta=(\sqrt{5}-1)/2 , \quad (2.3)$$

because of its extremely simple continued-fraction expansion

$$\zeta=[0,(1,)^\infty] . \quad (2.4)$$

Its convergents are $\zeta_n=F_n/F_{n+1}$, where F_n is the nth Fibonacci number that satisfies $F_{n+1}=F_n+F_{n-1}$ with $F_0=0$ and $F_1=1$. The golden-mean invariant circle is treated as the limiting case [13] of the set of periodic orbits with rotation numbers ζ_n.

The linear stability of a periodic orbit of rotation number p/q,

$$(r_0,\theta_0),(r_1,\theta_1),\ldots,(r_q,\theta_q)=(r_0,\theta_0+p) ,$$

is determined by the eigenvalues of the Jacobian matrix M of the q times iterated map at an orbit point, where

$$M=\prod_{i=0}^{q-1}\begin{bmatrix} b & kg_w'(\theta_i) \\ b & 1+kg_w'(\theta_i)\end{bmatrix} . \quad (2.5)$$

In the dissipative case ($0<b<1$), the eigenvalues depend on both the trace and the determinant of M:

$$\lambda=\{\mathrm{Tr}M\pm[(\mathrm{Tr}M)^2-4\,\mathrm{Det}M]^{1/2}\}/2 , \quad (2.6)$$

where $\mathrm{Det}M=b^q$. A quantity called the residue is defined by

$$R=(1+\mathrm{Det}M-\mathrm{Tr}M)/[2(1+\mathrm{Det}M)] . \quad (2.7)$$

For a stable orbit, $0<R<1$; for an unstable orbit, $R<0$ or $R>1$. Given the tuning parameter z and the dissipation parameter b, the residue of a periodic orbit depends on k and Ω.

The dissipative Wilbrink map has a periodic orbit of rotation number p/q in a region called the (p/q)-Arnol'd tongue [6] in the (Ω,k) space. For each value of k, the residue of this periodic orbit is changed as Ω is increased from the minimum value $\Omega_{\min}$ to the maximum value $\Omega_{\max}$, as shown in Fig. 1. At the two end points, $\Omega_{\min}$ and $\Omega_{\max}$, $R=0$. As Ω is increased from $\Omega_{\min}$, the residue increases and eventually reaches a maximum $R^*(k)$ at some $\Omega^*(k)$. As Ω is further increased beyond Ω^*, the residue decreases to zero.

Feigenbaum *et al.* [4], in their work on the dissipative standard map, found a criterion for the breakup of an invariant circle that is similar to Greene's residue criterion [13] for area-preserving maps. This criterion postulates a

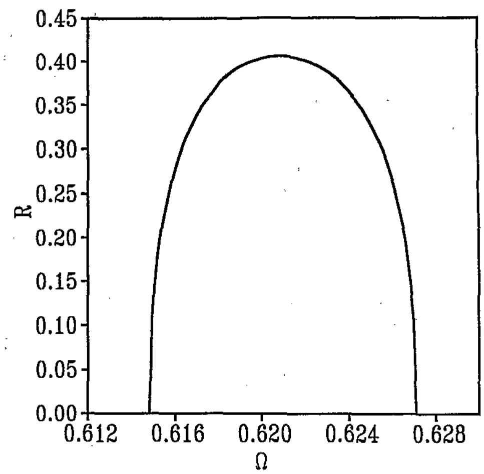

FIG. 1. The residue R as a function of Ω for the orbit with rotation number $\omega=\frac{5}{8}$ when $z=0.03$, $b=0.9$, and $k=0.8$.

close relation between the existence of an invariant circle with an irrational rotation number ω and the stability of the nearby periodic orbits whose rotation numbers p_n/q_n are the successive convergents of ω. For each value of k, one can easily obtain the maximum residue value $R_n^*(k)$ of the periodic orbit of rotation number p_n/q_n when $\Omega=\Omega_n^*(k)$. As $n\to\infty$, one finds numerically one of the following three cases.

(i) Subcritical case: $R_n^*\to 0$, and a smooth invariant circle exists.

(ii) Critical case: $R_n^*\to\frac{1}{2}$, and a nonsmooth continuous invariant circle exists.

(iii) Supercritical case: $R_n^*\to\infty$, and there is no invariant circle.

III. BREAKUP OF INVARIANT CIRCLES

Using the residue criterion for the analytic case when $z>0$, we study the breakup of the golden-mean invariant circle by varying the dissipation parameter from $b=1$ to 0.

Consider the case $z=0.03$. When $b=0.9$, the maximum residue $R^*(k)$ of a nearby periodic orbit is shown in Fig. 2(a). Unlike the dissipative standard map case, it is not a monotonic function of k. It tends to infinity right after the invariant circle has broken and becomes finite again as the invariant circle reappears. As shown in the figure, when $b=0.9$, there exist four bands which correspond to the stable regions in the k space where the invariant circle exists. These bands are separated by gaps which correspond to the unstable regions where the invariant circle is broken. Therefore, there exist seven critical points (three reappearance points k_r and four disappearance points k_d): $k_d^{(1)}=0.272\,313\,668$, $k_r^{(1)}=0.589\,595\,631$, $k_d^{(2)}=0.801\,067\,779$, $k_r^{(2)}=1.080\,967\,42$, $k_d^{(3)}=1.119\,771\,05$, $k_r^{(3)}=1.214\,671\,122$, and $k_d^{(4)}=1.248\,090\,589\,35$. Note that the number of critical points is the same as that for the area-preserving case ($b=1$) where the invariant circle reappears three times [7].

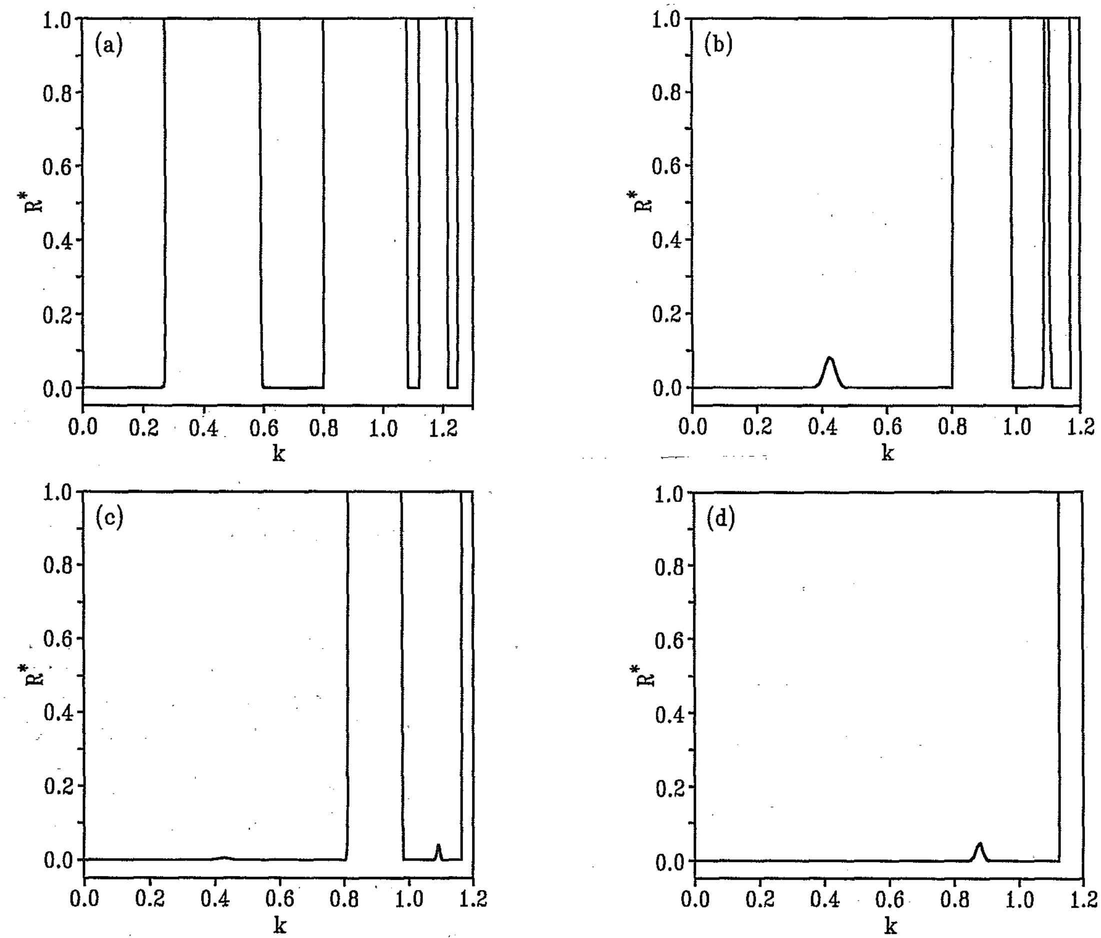

FIG. 2. The maximum residue R^* vs k plotted for the periodic orbit with rotation number $\omega=\frac{377}{610}$ for $z=0.03$: (a) $b=0.9$, (b) $b=0.75$, (c) $b=0.74$, and (d) $b=0.65$.

As the dissipation parameter is decreased from $b=0.9$, the first and second bands in Fig. 2(a) approach each other and merge into a band when $b\simeq0.75$. Therefore, when $b=0.75$, the number of bands becomes 3. This band-merging phenomenon is shown in Fig. 2(b). As b is further decreased from $b=0.75$, band-merging phenomena occur successively [see Figs. 2(c) and 2(d)] until a single band is left. Thus, as the dissipation parameter is decreased from $b=1$ to 0, the number of bands gradually decreases, and ultimately recurrence ceases at some dissipation parameter value (see Table I). Thus, when $b\leq0.65$, there is no reappearance of invariant circles since there exists only a single band. For this case, as b is decreased to zero, the critical points converge to $k_d=1$, which is just the critical value for the circle-map case when $b=0$.

We have studied the breakup of invariant circles for different values of z. As in the case $z=0.03$, the number of recurrences gradually decreases until a single band is left at a dissipation parameter value b^* dependent on z. The points of b^* lie on a curve in the (z,b) plane (see Fig. 3). Invariant circles can reappear in the region above the curve, whereas there is no recurrence of invariant circles in the region below the curve.

When $b^*<b<1$, the nearby resonances of the golden-mean invariant circle can be separate after they overlap. The invariant circle thus can reappear after its disappearance. As an example, consider two nearby resonances with rotation numbers $\omega=\frac{5}{8}$ and $\frac{8}{13}$ for $z=0.03$ and

TABLE I. Number of bands (N_b), number of disappearance points (N_d) and number of reappearance points (N_r) for various values of b when $z=0.03$.

b	N_b	N_d	N_r
1.0	4	4	3
0.9	4	4	3
0.8	4	4	3
0.76	4	4	3
0.75	3	3	2
0.74	2	2	1
0.66	2	2	1
0.65	1	1	0
0.5	1	1	0
0.3	1	1	0
0.1	1	1	0
0.0	1	1	0

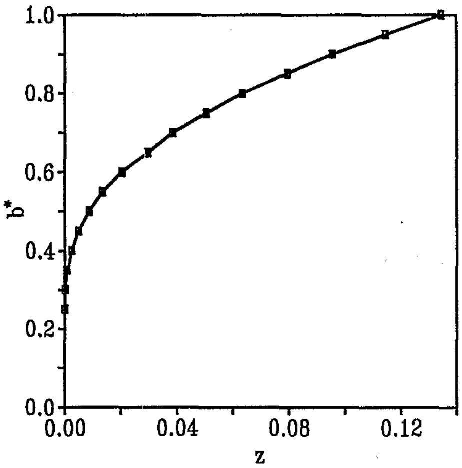

FIG. 3. The points b^* at which recurrence ceases are denoted by squares.

$b=0.9$. The Arnol'd tongues [6] of these resonances are shown in Fig. 4(a). There is an "hourglass" structure in the Arnol'd tongue. Unlike the case of the dissipative standard map, the widths of the resonances are not monotonic functions of k. This structure has previously been found in certain circle maps [14,15] where there is only a single quasiperiodic transition to chaos. Near the thin parts of the hourglass, the resonances separate after they overlap near the thick parts [see Fig. 4(a)]. There are three resonance-separating regions in the (Ω, k) space where the golden-mean invariant circle reappears after it has broken up.

As the dissipation parameter is decreased from $b=0.9$, the width of the resonance gradually decreases although the hourglass structure persists in the Arnol'd tongue [compare Fig. 4(a) with Fig. 4(c)]. Therefore, as b is decreased, the resonance-overlapping regions in Fig. 4(a) gradually disappear, and eventually there remains a single resonance-overlapping region [see Fig. 4(c)]. Thus, the number of recurrences gradually decreases as b is decreased, and when $b \le b^*$, there is no reappearance of invariant circles.

IV. SCALING

We will first summarize the definitions of the scaling factors [3–5]. The convergence rate of the sequence $\Omega_n^*(k)$ at which the residue of the periodic orbit with rotation number $\omega = F_n/F_{n+1}$ has its maximum value is defined to be

$$\delta_n(k) = \frac{\Omega_{n-1}^*(k) - \Omega_n^*(k)}{\Omega_n^*(k) - \Omega_{n+1}^*(k)} . \tag{4.1}$$

Let d_n be the angular distance between a point (θ_0, r_0) on a periodic orbit with $\omega = F_n/F_{n+1}$ and its nearest point

$$d_n = \theta_{F_n} - \theta_0 - F_{n-1} . \tag{4.2}$$

Define

$$\alpha_n(k) = \frac{d_{n-1}(k)}{d_n(k)} . \tag{4.3}$$

In the limit $n \to \infty$, $\delta_n(k)$ and $\alpha_n(k)$ converge asymptotically to the parameter scaling factor δ and the orbital scaling factor α.

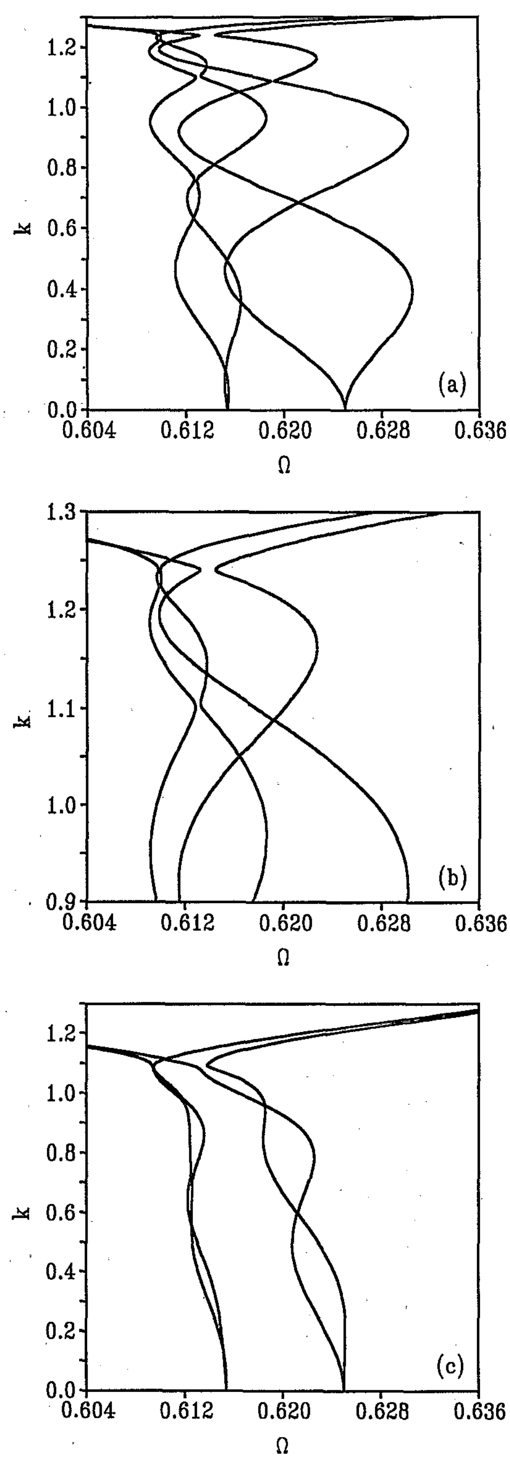

FIG. 4. Arnol'd tongues of resonances with rotation numbers $\omega = \frac{5}{8}$ (right one) and $\omega = \frac{8}{13}$ (left one) for $z=0.03$: (a) $b=0.9$, (b) a blowup of (a), and (c) $b=0.6$.

TABLE II. Values of δ_n, α_n, and the superconverged α_n^* when $z=0.03$, $b=0.9$, $k_d^{(1)}=0.272\,313\,668$.

n	δ_n	α_n	α_n^*
5	−2.634 85	−1.522 18	−1.558 42
6	−2.786 46	−1.601 45	−1.545 98
7	−2.840 36	−1.416 65	−1.373 03
8	−2.837 40	−1.381 36	−1.362 34
9	−2.837 98	−1.369 00	−1.382 89
10	−2.837 75	−1.256 85	−1.299 06
11	−2.837 60	−1.324 55	−1.292 22
12	−2.836 08	−1.262 68	−1.288 73
13	−2.835 11	−1.307 68	−1.287 79
14	−2.834 36	−1.272 03	−1.287 60
15	−2.834 00	−1.299 68	−1.287 71
16	−2.833 78	−1.278 56	−1.287 91
17	−2.833 70	−1.295 33	−1.288 10
18	−2.833 65	−1.282 63	−1.288 26
19	−2.833 63	−1.292 73	−1.288 37
20	−2.833 62	−1.285 05	−1.288 44

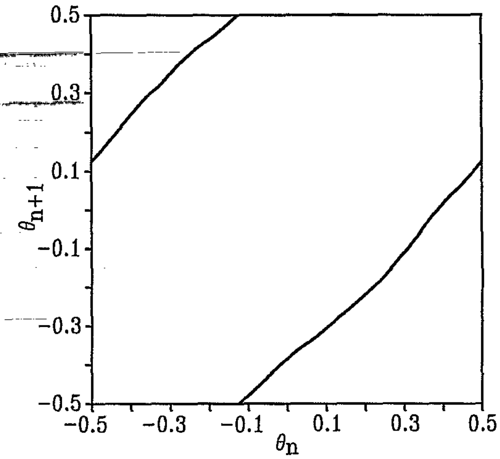

FIG. 5. A critical reduced map when $z=0.03$, $b=0.9$, $k=k_d^{(1)}=0.272\,313\,668$, and $\Omega=0.617\,209\,846$.

To look for orbital scaling behavior, a convenient orbit point θ_0 must be chosen. For example, in the sine-circle map, the origin ($\theta_0=0$) is a convenient point near which simple scaling is found [3]. Note that in this case the region near $\theta_0=0$ is the most rarified. However, in the case of the dissipative Wilbrink map, the origin is not the most convenient point to look for simple scaling since in general the region near the origin is not the most rarified. The most rarified region depends on z, b, and k_c (critical point). In all the cases studied, we found simple scaling in the most rarified region. The most rarified region can be easily located by choosing an orbit point which has the largest angular distance d_n from its nearest orbit point.

We have calculated δ and α in the most rarified region at the critical points of disappearance and reappearance for many values of b and z. For all the cases studied, it was found that the values of δ and α are equal and the same as those in the circle maps with a cubic inflection point [3–5]. Consider, for example, the case $z=0.03$, $b=0.9$, and $k_d^{(1)}=0.272\,313\,668$. In Table II we listed the values of δ_n and α_n. Note that the convergence of α_n is slower than that of δ_n. To improve the convergence of the α_n sequence, we follow MacKay [16] by superconverging the sequence and thus obtain a new α_n^* sequence which has the same limit (see Table II). The values of δ_n and α_n^* converge to the scaling factors δ and α of the circle maps with a cubic inflection point.

From the point of view of universality, the good agreement of the scaling factors with those in circle maps with a cubic inflection point is a surprise since circle maps induced on invariant circles have no inflection points. Instead of looking at the original two-dimensional map, we follow Bohr *et al.* [6] by projecting out the angle variable θ and considering the corresponding one-dimensional circle map [$\theta_{n+1}=f(\theta_n)$], called a "reduced map." A critical reduced map is shown in Fig. 5 for $z=0.03$ and $b=0.9$. The derivative of the reduced map has its minimum value $f'(\theta)=0.66$ when $\theta=0.048$. Therefore, the critical reduced map does not have any inflection point. In fact, for all the cases studied, the critical reduced map has no inflection point, like the coupled logistic map case [17]. Therefore, although the order of an inflection point serves as one of the universality criteria in circle maps [5,18–20], it is no longer a criterion to determine the universality class in two-dimensional dissipative maps.

V. SUMMARY

We have studied the dissipative Wilbrink map with a tuning parameter z for the analytic case when $z>0$. For small z values and weakly dissipative cases, the golden-mean invariant circle experiences a sequence of disappearance and reappearance as the nonlinearity is increased. This recurrence phenomenon takes place as the result of the separation of the nearby resonances after they overlap. As the dissipation is increased, the number of recurrences gradually decreases due to the merging of bands. Ultimately, a single band is left at a dissipation parameter value b^* dependent on z. Thus there is no reappearance of invariant circles when $b \le b^*$. The scaling factors at the disappearance and reappearance points are the same as those for the circle map with a cubic inflection point although the critical reduced map has no inflection point.

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