Scaling pattern of period doubling in four dimensions

Sang-Yoon Kim
Department of Physics, Kangwon National University, Chunchon, Kangwon-Do 200-701, Korea

Bambi Hu
Department of Physics, University of Houston, Houston, Texas 77204-5504
(Received 17 July 1989; revised manuscript received 1 December 1989)

The period-M ($M=1$ and 2) scaling pattern of period doubling in a symmetric four-dimensional volume-preserving map is studied. The period-doubling sequence repeats itself asymptotically from one bifurcation to the next in a period-1 bifurcation route, and to every other one in a period-2 bifurcation route. The parameter-scaling factors γ_1 and γ_2 in a bifurcation route depend on the bifurcation path. They are some combination of δ_1 and δ_2 (divergence rates from the period-M map of the renormalization transformation) and δ'_1 and δ'_2 (convergence rates to the period-M map). Therefore each bifurcation route is characterized by these four scaling factors. The values of δ_1, δ_2, δ'_1, and δ'_2 are obtained by a numerical calculation and a renormalization analysis. We find that there are three kinds of period-1 scaling patterns and one kind of period-2 scaling pattern. δ_1 and δ'_1 in any bifurcation route are the same as those in area-preserving maps; however, δ_2 and δ'_2 depend on the bifurcation route.

I. INTRODUCTION

The discovery of universal scaling behavior of period doubling in one-dimensional (1D) maps by Feigenbaum[1] inspired several authors[2–11] to study the scaling behavior of period doubling in area-preserving maps. Although the universality results for period doubling in 1D maps extend to higher-dimensional dissipative maps, the scaling behavior of area-preserving maps is distinctly different. An interesting question is whether the self-similar period-doubling pattern of area-preserving maps carries over to higher-dimensional conservative maps. Consequently, period doubling in four-dimensional (4D) symplectic maps has been studied.[9,12] Clear evidence for an infinite period-doubling sequence in a 4D symplectic map has been reported by Mao, Satija, and Hu.[13] The infinite period-doubling sequence was determined by following a special bifurcation path. Many additional bifurcation paths and their scaling behavior in symmetric 4D volume-preserving maps have been found.[14,15] However, only the period-1 scaling pattern has been studied.

The purpose of this paper is to report on the period-M ($M=1$ and 2) scaling pattern of period doubling in a symmetric 4D volume-preserving map. By generalizing the concept of a bifurcation route and a bifurcation path, we find that there are infinite kinds of bifurcation routes. In this paper, we study the scaling pattern in the period-M ($M=1$ and 2) bifurcation route. The period-doubling sequence repeats itself asymptotically from one bifurcation to the next in a period-1 bifurcation route, and to every other one in a period-2 bifurcation route. The parameter-scaling factors γ_1 and γ_2 (γ_1^{-1} and γ_2^{-1} are the period-doubling convergence rates) in a bifurcation route depend on the bifurcation path. Their values for "almost all" bifurcation paths (called "regular" paths) in a bifurcation route are different from those for exceptional bifurcation paths (called "special" paths). However, it is shown that the two parameter-scaling factors for any (regular or special) bifurcation path are some combination of the four scaling factors δ_1 and δ_2 (divergence rates from the period-M map of the renormalization transformation) and δ'_i and δ'_2 (convergence rates to the period-M map). The divergence rates δ_1 and δ_2 are the parameter-scaling factors γ_1 and γ_2 for regular paths, and δ'_1 and δ'_2 are the rates at which the stability indices[18] of the daughter orbits converge geometrically to the critical stability indices. Hereafter, we call δ_1, δ_2, δ'_1, and δ'_2 the "fundamental noncoordinate scaling factors," in the sense that the two parameter-scaling factors for any bifurcation path can be expressed in terms of them. Therefore each bifurcation route is characterized by its own four fundamental noncoordinate scaling factors. The values of these four fundamental noncoordinate scaling factors are obtained by a direct numerical study and a renormalization method. We find that there are three kinds of period-1 bifurcation routes and one kind of period-2 bifurcation route.

This paper is organized as follows. We begin by recapitulating some useful properties of a symmetric 4D volume-preserving map in Sec. II. We then generalize the concept of a bifurcation route and a bifurcation path. We also introduce the "route sequence" of a bifurcation route to classify all the scaling patterns of period doubling. In Sec. III the scaling patterns of all three kinds of period-1 bifurcation routes are given. We obtain in Sec. IV the fundamental noncoordinate scaling factors for the period-2 scaling pattern. Section V gives a summary.

II. BIFURCATION ROUTE, BIFURCATION PATH, AND ROUTE SEQUENCE

We review some useful properties of a symmetric 4D volume-preserving map in the Sec. II A. We then generalize in Secs. II B and II C the concept of a bifurcation route and a bifurcation path. In Sec. II D, we define the "route sequence" of a bifurcation route.

A. Symmetric 4D volume-preserving map

We study the period-M scaling pattern in a symmetric 4D volume-preserving map. The symmetric 4D volume-preserving map T is of the following form:

$$T: \begin{cases} x' = -y + f(x,u) , \\ y' = x , \\ u' = -v + g(x,u) , \\ v' = u , \end{cases} \tag{2.1}$$

where $f(x,u)=2[(Cx+x^2)+E(u+Fu^2+Gxu)]$ and $g(x,u)=f(u,x)$. The term $Cx+x^2$ in $f(x,u)$ is a quadratic function of x, and C a parameter. The term $u+Fu^2+Gxu$ in $f(x,u)$ contains all the couping terms up to quadratic terms. E is their common coefficient, called the coupling parameter. F and G are parameters; however, we will fix their values to perform a two-parameter search.[13] Since $g(x,u)=f(u,x)$, the map T (2.1) is called "symmetric."[14]

There are two kinds of orbits in the map T.[14] One is the "in-phase" orbit,

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \quad i=1,\ldots,N$$

where N is the orbit. The other one is the "opposite-phase" orbit,

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} x_{i+N/2} \\ y_{i+N/2} \end{bmatrix} .$$

In this paper, we consider only the in-phase orbit.

For the in-phase orbit, the Jacobian matrix $\underline{L}$ of the map (2.1) decomposes into two 2×2 matrices under a coordinate transformation:[14]

$$\begin{aligned} X &= \frac{R(x+u)}{2} , \\ Y &= \frac{R(y+v)}{2} , \\ U &= \frac{R(x-u)}{2} , \\ V &= \frac{R(y-v)}{2} , \end{aligned} \tag{2.2}$$

where $R=1+EF+EG$. T thus becomes

$$T: \begin{cases} X' = -Y + F(X,U) , \\ Y' = X , \\ U' = -V + G(X,U) , \\ V' = U , \end{cases} \tag{2.3}$$

where

$$\begin{aligned} F(X,U) &= 2[PX+X^2+F_1U^2] , \\ G(X,U) &= 2[(P-2E)U+G_1XU] , \\ P &= C+E , \\ F_1 &= (1+EF-EG)/R , \\ G_1 &= 2(1-EH) , \\ H &= (2F+G)/R . \end{aligned}$$

Then, the in-phase orbit of the old map (2.1) becomes the orbit of the new map with $U=V=0$. Moreover, the new coordinates (X,Y) give the 2D Hénon map,

$$\begin{aligned} X' &= -Y + 2(PX+X^2), \\ Y' &= X . \end{aligned} \tag{2.4}$$

The Jacobian matrix $\underline{L}$ of the new map at the in-phase orbit can be decomposed:

$$\underline{L} = \begin{bmatrix} \underline{L}_1 & \underline{0} \\ \underline{0} & \underline{L}_2 \end{bmatrix} , \tag{2.5}$$

where $\underline{0}$ is the 2×2 null matrix, and

$$\underline{L}_1 = \begin{bmatrix} 2P+4X & -1 \\ 1 & 0 \end{bmatrix} ,$$

$$\underline{L}_2 = \begin{bmatrix} 2(P-2E)+2G_1X & -1 \\ 1 & 0 \end{bmatrix} .$$

Here the matrix $\underline{L}_1$ is just the Jacobian matrix of the 2D Hénon map (2.3).

The map T (2.1) is symplectic[14] only if

$$\frac{\partial f}{\partial u} = \frac{\partial g}{\partial x} . \tag{2.6}$$

The stability of an orbit of period N in a 4D symplectic map is determined by the Jacobian matrix $\underline{M}$ of T^N which is symplectic. As is well known,[16] if λ is an eigenvalue of $\underline{M}$, so are λ^{-1} and λ^*. Therefore the eigenvalues λ's come either in reciprocal pairs which are real or of modulus unity, or in a complex quadruplet with $\lambda_1=\lambda_2^{-1}=\lambda_3^*=\lambda_4^{*-1}$. These eigenvalues of $\underline{M}$ are called the multipliers of the orbit.[9] Following Broucke,[17] Howard and MacKay[18] associate with each eigenvalue λ a stability index

$$\rho = \lambda + \lambda^{-1} . \tag{2.7}$$

Then, the reduced characteristic polynomial of a 4D symplectic matrix is quadratic:[18]

$$\rho^2 - T_1\rho + T_2 - 2 = 0 , \tag{2.8}$$

where

$$T_1 = \sum_{i=1}^{4} \lambda_i = \mathrm{Tr}\underline{M} = \rho_1 + \rho_2 ,$$

$$T_2 = \sum_{\substack{i,j \\ (i<j)}} \lambda_i\lambda_j = [(\mathrm{Tr}\underline{M})^2 - \mathrm{Tr}(\underline{M}^2)]/2 = \rho_1\rho_2 + 2 .$$

Therefore the two independent quantities (T_1, T_2) or

(ρ_1,ρ_2) determine the stability of the periodic orbit.[17,18] A periodic orbit is spectrally stable if and only if all stability indices are real with $|\rho|\leq 2$, and a period-doubling bifurcation occurs when two eigenvalues coalesce at $\lambda=-1$ and split along the negative real axis (a stability index decreases through -2).[18]

The map T (2.1) is a volume-preserving map since $\mathrm{Det}(\underline{L})=1$; it is symplectic only if $G=2F$. However, for the in-phase orbit, Eq. (2.6) is always satisfied because of the symmetry of the map $g(x,u)=f(u,x)$. Therefore the stability diagram in the T_1-T_2 plane for the orbits in a 4D symplectic map is the same as that for the in-phase orbit in a symmetric 4D volume-preserving map.[14] Since the Jacobian matrix of the new map (2.3) at the in-phase orbit is decomposed as shown in (2.5), the stability index ρ_1 of a periodic orbit is a function of only one parameter P, and ρ_2 a function of the two parameters P and E. We fix the values of F and G to perform a two-parameter search:[13]

$$\rho_1=\rho_1(P)\ ,\qquad \rho_2=\rho_2(P,E)\ . \tag{2.9}$$

B. Bifurcation route

A remarkable observation of Mao and Helleman[14] is that a "mother" stability region bifurcates into two "daughter" stability regions in the parameter plane as shown in Fig. 1. Therefore the stability diagram in the parameter plane can be regarded as a kind of "binary tree." We denote the upper branch of the two daughter stability regions by U, and the lower branch by L. We call the direction of the upper branch, "U direction"; the direction of the lower branch, "L direction." A bifurcation route is then uniquely determined by its address, which is an infinite sequence of U and L. Therefore there are infinite kinds of bifurcation routes.

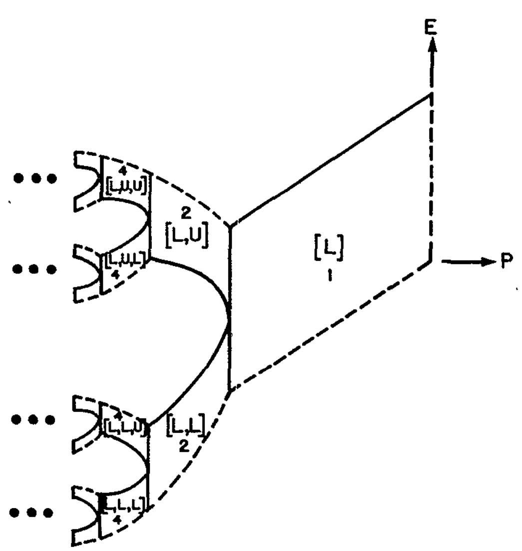

FIG. 1. A schematic stability diagram in the PE-parameter plane for the period-1, -2, -4, etc., orbits of the map (2.3). The period-doubling bifurcation line is denoted by the solid line, and the tangent bifurcation line is denoted by the dashed line.

A bifurcation route is called "period M" (M a positive integer) if the period-doubling sequence in it exhibits asymptotically a period-M scaling pattern. In this paper, we consider only period-M ($M=1$ and 2) bifurcation routes. It is found that there are three kinds of period-1 bifurcation routes and one kind of period-2 bifurcation route. The three kinds of period-1 bifurcation routes are as follows. The first one is the "S route," whose address is $[a,(U,)^\infty]$ or $[b,(L,)^\infty]$, a and b being arbitrary finite sequences. An S route is formed if one follows asymptotically only the upper branch or the lower branch. Since one goes asymptotically in the same direction (U or L direction), we call it an S route. The second one is an "A route," whose address is $[c,(L,U,)]$. Since the address of an A route is $[c,(L,U,)^\infty]$, the direction of the route asymptotically alternates between the L direction and the U direction. Therefore we call it an A route. The third one is the "E route," whose address is $[(L,U,)^\infty]$. Since the address is unique, there is only one E route. The differences between an A route and the E route are as follows. The value of the coupling parameter E^* at the accumulation point (P^*,E^*) in the E route is zero, whereas in any A route it is nonzero. In these three kinds of period-1 bifurcation routes, the period-doubling sequence exhibits a period-1 scaling pattern. As mentioned above, there is only one kind of period-2 bifurcation route in which the period-doubling sequence exhibits asymptotically a period-2 scaling pattern. The address of a period-2 bifurcation route is $[d,(L,L,U,U,)^\infty]$.

C. Bifurcation path

In this section we define the period-M bifurcation path and compare our period-1 bifurcation path with those previously defined in Ref. 14.

Before defining a bifurcation path, we explain some terms and notations that will be used later. We call an orbit born by the nth period-doubling bifurcation in the map (2.3) an orbit of level n. Then, the period N of an orbit of level n is 2^n, and there are 2^n orbits of level n. As explained in Sec. II A, the stability of an orbit of level n is determined by its multipliers $(\lambda_{1,n},\lambda_{2,n},\lambda_{3,n},\lambda_{4,n})$, its stability indices $(\rho_{1,n},\rho_{2,n})$, or $(T_{1,n},T_{2,n})$. The values of the two parameters P and E in the map (2.3) at which an orbit of level n has some given multipliers, or, equivalently, stability indices, will be denoted by P_n and E_n.

Let us choose a period-1 bifurcation route. Then, a period-1 bifurcation path which belongs to the chosen bifurcation route is formed by following in the chosen bifurcation route P_n and E_n at which the orbit of level n has some given multipliers $(\lambda_1,\lambda_1^{-1},e^{i\theta},e^{-i\theta})$ or stability indices (ρ_1,ρ_2), where

$$\begin{aligned}&\lambda_1 \text{ is any real number (i.e., } \lambda_1\in\mathbb{R})\ ,\\ &0\leq\theta\leq\pi\ ;\\ &\rho_1\in\mathbb{R}\ ,\\ &|\rho_2=2\cos\theta|\leq 2\ .\end{aligned} \tag{2.10}$$

Similarly, a period-2 bifurcation route is defined as follows. Choose a period-2 bifurcation route whose address is $[d,(L,L,U,U,)^{\infty}]$. Then, a period-2 bifurcation path that belongs to the chosen bifurcation route is formed by following in the chosen bifurcation route P_n and E_n at which the orbit of level n has some given multipliers $(\lambda_1,\lambda_1^{-1},e^{i\theta_1},e^{-i\theta_1})$ for n even, and $(\lambda_1,\lambda_1^{-1},e^{i\theta_2}, e^{-i\theta_2})$ for n odd, where

$$\lambda_1 \in \mathbb{R} ,\quad 0\le\theta_1,\theta_2\le\pi . \tag{2.11}$$

The period-1 scaling pattern has been studied previously,[14] and three kinds of bifurcation routes were found ("L route," "U route," and "E route"). We compare our period-1 bifurcation route with that found by Mao and Helleman. Their bifurcation path is formed by following asymptotically P_n and E_n at which the orbit of level n has some given multiplier $(-1,-1,e^{i\theta},e^{-i\theta})$, which correspond to a bifurcation point lying on the period-doubling bifurcation line in the T_1-T_2 plane. A bifurcation path with $0\le\theta\le\pi/2$ is called an L_θ path (or L path if θ is not specified).[14] Then, an L route is formed by a particular L path and all L_θ paths in its neighborhood. Its address is $[a,(U,)^{\infty}]$ or $[b,(L,)^{\infty}]$, which is the same as that of our S route. However, they considered only the case $\lambda_1=-1$ and $0\le\theta\le\pi/2$, whereas we consider the case $\lambda_1\in\mathbb{R}$ and $0\le\theta\le\pi$ in the S route [see Eq. (2.10)]. In this sense, their L route is a proper subset of our S route. Similarly, it is easy to see that their U and E routes are proper subsets of our A and E routes, respectively.

D. Route sequence

In this section we define the "route sequence" of a bifurcation route. The route sequence of a bifurcation route is uniquely determined by its address as follows. If the nth element in its address is the same as the next $(n+1)$th element, we assign a 0 to the nth element of the route sequence; otherwise we assign a 1. Therefore the route-sequence is an infinite sequence of the two numbers 0 and 1.

The route sequence of the period-M bifurcation route is as follows. We first consider the period-1 bifurcation route. The route sequence of an S route is $[a',(0,)^{\infty}]$ since one goes asymptotically in the same direction (U or L direction) in the S route. On the other hand, the route sequence of an A or E route is $[b',(1,)^{\infty}]$, since the direction of the A or E route alternates between the L direction and the U direction. Here, b' is a finite nonempty sequence for an A route, whereas it is empty for the E route. Note that the route sequence of any period-1 bifurcation route exhibits eventually a period-1 pattern. Second, let us consider a period-2 bifurcation route whose address is $[c,(L,L,U,U,)^{\infty}]$. Then the route sequence of the period-2 bifurcation route is $[c',(0,1,)^{\infty}]$. Note also that the route sequence of any period-2 bifurcation route exhibits a period-2 pattern. From the period-M pattern of the route sequence for the period-M bifurcation route, it may be conjectured that in a bifurcation route whose route sequence exhibits a period-M (M any positive integer) behavior, the period-doubling sequence exhibits asymptotically a period-M scaling pattern, and, in a bifurcation route whose route sequence is random, it exhibits a "chaotic" scaling pattern.

III. PERIOD-1 SCALING PATTERNS

In this section the results of period-1 scaling pattern in the three kinds of period-1 bifurcation routes (S, A, and E routes) are given. In Sec. III A, we obtain the parameter-scaling factors γ_1 and γ_2 by the scaling-matrix method.[19] The values of γ_1 and γ_2 depend on the period-1 bifurcation path, and it is found that there are more "special" bifurcation paths than those found in Ref. 14. It is shown in Sec. III B that the two parameter-scaling factors for any bifurcation path in a bifurcation can be expressed in terms of the four "fundamental non-coordinate scaling factors" of the bifurcation route, δ_1, δ_2 δ'_1, and δ'_2. In Sec. III C, we review the orbital scaling behaviors which have been studied in Ref. 14.

A. Parameter-scaling factor

To perform a two-parameter search,[13] we consider the case that the values of (F,G) are (1,2), (2,4), (1,3), and (2,3) and follow the orbit up to level 17. The parameter-scaling factors are independent of the values of F and G within numerical accuracy. This is expected as we are considering a codimension-two problem.

We first define a "regular" path and a "special" path as follows. Choose a period-1 bifurcation route. Then, for any period-1 bifurcation path that belongs to the chosen bifurcation route, (P_n,E_n) converges to the same accumulation ponts (P^*,E^*)

$$\lim_{n\to\infty}(P_n,E_n)=(P^*,E^*) \tag{3.1}$$

for all bifurcation paths. Furthermore, at the accumulation point (P^*,E^*), the stability indices $\rho_{1,n}$ and $\rho_{2,n}$ converge geometrically to the critical stability indices ρ_1 and ρ_2, respectively,

$$\lim_{n\to\infty}\rho_{1,n}(P^*)=\rho_1^* ,\quad \lim_{n\to\infty}\rho_{2,n}(P^*,E^*)=\rho_2^* . \tag{3.2}$$

Note that $\rho_{1,n}$ is a function of only one parameter P [see Eq. (2.9)]. The critical stability indices are shown in Table I. If the given values of stability indices ρ_1 and ρ_2 in Eq. (2.10) are not the critical values ($\rho_1\neq\rho_1^*$ and $\rho_2\neq\rho_2^*$), then we call it a "regular" path; otherwise, we call it a "special" path.

The scaling behavior of the period-doubling sequence $[(P_n,E_n),n=0,1,2,\ldots]$ can be determined by the scaling-matrix method developed by Guckenheimer, Hu, and Rudnick[19] (refer to Ref. 13 for details). The 2×2 scaling matrix of level n, $\underline{\Gamma}_n$, is defined as follows:

$$\begin{bmatrix} P_n-P_{n-1} \\ E_n-E_{n-1} \end{bmatrix} = \underline{\Gamma}_n \begin{bmatrix} P_{n+1}-P_n \\ E_{n+1}-E_n \end{bmatrix} . \tag{3.3}$$

TABLE I. The critical stability indices ρ_1^* and ρ_2^* in a period-1 bifurcation route.

Route	ρ_1^*	ρ_2^*
S	$-2.543\,510\,20$	$2.000\,000\,00$
A	$-2.543\,510\,20$	$-1.000\,000\,00$
E	$-2.543\,510\,20$	$-2.543\,510\,20$

Then, $\underline{\Gamma}_n$ approaches a constant matrix $\underline{\Gamma}$ as $n \to \infty$:

$$\lim_{n\to\infty} \underline{\Gamma}_n = \underline{\Gamma} \; . \tag{3.4}$$

The eigenvalues of $\underline{\Gamma}$, γ_1, and γ_2, are just the parameter-scaling factors. The parameter-scaling factors in the three kinds of period-1 bifurcation routes are shown in Table II. The values of γ_1 and γ_2 in each bifurcation route depend on the bifurcation path. Regular paths have the same values of γ_1 and γ_2, whereas each kind of special path has different values of γ_1 and γ_2 from those of regular paths. In the S and A routes, there are three kinds of special paths. On the other hand, in the E route, there are two kinds of special paths since ρ_2 for any bifurcation path [see the range of ρ_2 in Eq. (2.10)] cannot be ρ_2^* (see Table I). No γ_2 exists for the first type of special path in the E route, since E_n is zero for all n. In Ref. 14, they found only one special path in each bifurcation route (L, U, and E route) which belongs to the first type of special path in each bifurcation route (S, A, and E route). Therefore, by generalizing the bifurcation route and the bifurcation path as explained in Sec. II, we find that there are more special bifurcation paths than those found in Ref. 14.

B. Fundamental noncoordinate scaling factor

In this section we find that there are four "fundamental noncoordinate scaling factors," δ_1 and δ_2 (divergence rates from the fixed map of the renormalization transformation) and δ_1' and δ_2' (convergence rates to the fixed map).

At the accumulation point (P^*,E^*) in a bifurcation route, the stability indices $\rho_{1,n}(P^*)$ and $\rho_{2,n}(P^*,E^*)$ converge to the critical stability indices ρ_1^* and ρ_2^*, respectively. The convergence is asymptotically geometric at rates δ_1' and δ_2', respectively,

$$\begin{aligned} &\rho_{1,n}(P^*)-\rho_1^* \sim \delta_1'^{\,n} \; , \\ &\rho_{2,n}(P^*,E^*)-\rho_2^* \sim \delta_2'^{\,n} \; . \end{aligned} \tag{3.5}$$

This implies that the fixed map has two stable directions under the renormalization transformation with eigenvalues δ_1' and δ_2'. Therefore δ' and δ' are convergence rates to the fixed map. The values of δ_1' and δ_2' are shown in Table III. Since $|\delta_2'| \ge |\delta_1'|$ (equality holds only for the E-route), δ_2' is the "essential" convergence rate which is the largest noncoordinate eigenvalue inside the unit circle.[9]

By comparing the analytic formulas of δ_1 and δ_2 (divergence rates from the fixed map) with those of the parameter-scaling factors γ_1 and γ_2, we show that δ_1, δ_2, δ_1', and δ_2' are "fundamental noncoordinate scaling factors." First, we obtain the analytic formulas for δ_1 and δ_2 by using the eigenvalue-matching renormalization method.[20] The basic idea of Derrida *et al.*[20] is to associate, for each (P,E), a value (P',E') such that $T^{(n+1)}_{(P',E')}$ locally resembles $T^{(n)}_{(P,E)}$; $T^{(n)}$ is the 2^nth iterated map of T (i.e., $T^{(n)}=T^{2^n}$). An approximate way to do it is to equate the stability indices of level n, $\rho_{1,n}(P,E)$ and $\rho_{2,n}(P,E)$, to those of level $(n+1)$, $\rho_{1,n+1}(P',E')$ and $\rho_{2,n+1}(P',E')$,

$$\begin{aligned} &\rho_{1,n}(P,E)=\rho_{1,n+1}(P',E') \\ &\rho_{2,n}(P,E)=\rho_{2,n+1}(P',E') \; . \end{aligned} \tag{3.6}$$

The accumulation point (P^*,E^*) is a fixed point of the recurrence relation (3.6),

$$\begin{aligned} &\rho_{1,n}(P^*,E^*)=\rho_{1,n+1}(P^*,E^*) \; , \\ &\rho_{2,n}(P^*,E^*)=\rho_{2,n+1}(P^*,E^*) \; . \end{aligned} \tag{3.7}$$

By linearizing Eq. (3.7) about the accumulation point (P^*,E^*), we obtain

$$\begin{aligned} \begin{bmatrix} \Delta P \\ \Delta E \end{bmatrix} &= \begin{bmatrix} \left.\dfrac{\partial P}{\partial P'}\right|_* & \left.\dfrac{\partial P}{\partial E'}\right|_* \\ \left.\dfrac{\partial E}{\partial P'}\right|_* & \left.\dfrac{\partial E}{\partial E'}\right|_* \end{bmatrix} \begin{bmatrix} \Delta P' \\ \Delta E' \end{bmatrix} \\ &= \Delta_n \begin{bmatrix} \Delta P' \\ \Delta E' \end{bmatrix} , \end{aligned} \tag{3.8}$$

TABLE II. The parameter-scaling factors γ_1 and γ_2 for a period-1 bifurcation path in a period-1 bifurcation route. In the second column, we denote a regular path by R and a special path by S. The ranges of ρ_1 and ρ_2 are given in Eq. (2.10).

Route	Path	γ_1	γ_2
S	$\rho_1\neq\rho_1^*$, $\rho_2\neq\rho_2^*$ (R)	8.721	4.000
	$\rho_1\neq\rho_1^*$, $\rho_2=\rho_2^*$ (S)	8.721	-15.08
	$\rho_1=\rho_1^*$, $\rho_2\neq\rho_2^*$ (S)	-74.78	4.000
	$\rho_1=\rho_1^*$, $\rho_2=\rho_2^*$ (S)	-74.78	-15.08
A	$\rho_1\neq\rho_1^*$, $\rho_2\neq\rho_2^*$ (R)	8.721	-2.000
	$\rho_1\neq\rho_1^*$, $\rho_2=\rho_2^*$ (S)	8.721	-15.08
	$\rho_1=\rho_1^*$, $\rho_2\neq\rho_2^*$ (S)	-74.78	-2.000
	$\rho_1=\rho_1^*$, $\rho_2=\rho_2^*$ (S)	-74.78	-15.08
E	$\rho_1\neq\rho_1^*$ (R)	8.721	-4.404
	$\rho_1=\rho_2\in[-2,2]$ (S)	8.721	nonexistent
	$\rho_1=\rho_1^*$ (S)	-74.78	-4.404

TABLE III. The four fundamental noncoordinate scaling factors δ_1, δ_2, δ_1' and δ_2' in a period-1 bifurcation route.

Route	δ_1	δ_2	δ_1'	δ_2'
S	8.721	4.000	-0.1166	-0.2653
A	8.721	-2.000	-0.1166	0.1326
E	8.721	-4.404	-0.1166	-0.1166

where $\Delta P = P - P^*$, $\Delta E = E - E^*$, $\Delta P' = P' - P^*$, $\Delta E' = E' - E^*$, and

$$\underline{\Delta}_n = \underline{A}_n^{-1} \underline{B}_n ,$$

$$\underline{A}_n = \begin{bmatrix} \left.\frac{\partial \rho_{1,n}}{\partial P}\right|_* & \left.\frac{\partial \rho_{1,n}}{\partial E}\right|_* \\ \left.\frac{\partial \rho_{2,n}}{\partial P}\right|_* & \left.\frac{\partial \rho_{2,n}}{\partial E}\right|_* \end{bmatrix},$$

$$\underline{B}_n = \begin{bmatrix} \left.\frac{\partial \rho_{1,n+1}}{\partial P'}\right|_* & \left.\frac{\partial \rho_{1,n+1}}{\partial E'}\right|_* \\ \left.\frac{\partial \rho_{2,n+1}}{\partial P'}\right|_* & \left.\frac{\partial \rho_{2,n+1}}{\partial E'}\right|_* \end{bmatrix},$$

where the asterisk denotes the accumulation point (P^*, E^*). Then, the eigenvalues $\delta_1^{(n)}$ and $\delta_2^{(n)}$ of the matrix $\underline{\Delta}_n$ are

$$\delta^{(n)} = \frac{\mathrm{Tr}\underline{\Delta}_n \pm [(\mathrm{Tr}\underline{\Delta}_n)^2 - 4\,\mathrm{Det}(\underline{\Delta}_n)]^{1/2}}{2} \quad (3.9)$$

Note that $\rho_1 = \rho_1(P)$ and $\rho_2 = \rho_2(P,E)$ in the map (2.3) [see Eq. (2.9)]. After some algebra, we obtain the analytic formulas for $\delta_1^{(n)}$ and $\delta_2^{(n)}$:

$$\delta_1^{(n)} = \frac{\left.\frac{d\rho_{1,n+1}}{dP'}\right|_*}{\left.\frac{d\rho_{1,n}}{dP}\right|_*}, \quad \delta_2^{(n)} = \frac{\left.\frac{\partial\rho_{1,n+1}}{\partial E'}\right|_*}{\left.\frac{\partial\rho_{1,n}}{\partial E}\right|_*}. \quad (3.10)$$

As $n \to \infty$, $\delta_1^{(n)}$ and $\delta_2^{(n)}$ approach δ_1 and δ_2, which are just the divergence rates from the fixed map,

$$\delta_1 = \lim_{n\to\infty} \delta_i^{(n)}, \quad i = 1,2 . \quad (3.11)$$

Secondly, we obtain the analytic formulas of γ_1 and γ_2 (parameter-scaling factors) by using the scaling-matrix method.[19] Note that a bifurcation path is formed by following in the chosen bifurcation route (P_n, E_n) at which the orbit at level n has some given stability indices ρ_1 and ρ_2 [see Eq. (2.10)]. Let us denote the given stability indices ρ_1 and ρ_2 by $G\rho_1$ and $G\rho_2$ and write them in the following form:

$$G\rho_1 = \rho_1^* + \Delta\rho_1 , \quad G\rho_2 = \rho_2^* + \Delta\rho_2 . \quad (3.12)$$

Then, by the definition of a bifurcation path, we obtain

$$\begin{aligned} G\rho_1 &= \rho_1^* + \Delta\rho_1 = \rho_{1,n}(P_n, E_n) , \\ G\rho_1 &= \rho_1^* + \Delta\rho_1 = \rho_{1,n+1}(P_{n+1}, E_{n+1}) , \\ G\rho_2 &= \rho_2^* + \Delta\rho_2 = \rho_{2,n}(P_n, E_n) , \\ G\rho_2 &= \rho_2^* + \Delta\rho_2 = \rho_{2,n+1}(P_{n+1}, E_{n+1}) . \end{aligned} \quad (3.13)$$

By linearizing Eq. (3.13) about the accumulation point (P^*, E^*) and using Eq. (3.5), we obtain

$$\begin{bmatrix} \Delta P_n \\ \Delta E_n \end{bmatrix} = \underline{\Gamma}_n \begin{bmatrix} \Delta P_{n+1} \\ \Delta E_{n+1} \end{bmatrix}, \quad (3.14)$$

where $\Delta P_n = P_n - P^*$, $\Delta E_n = E_n - E^*$, and $\underline{\Gamma}_n = \underline{A}_n^{-1} \underline{C}_n$, where $\underline{A}_n$ is defined in Eq. (3.8), and $\underline{C}_n$ depends on the values of $G\rho_1$ and $G\rho_2$ as follows.

(1) $G\rho_1 \neq \rho_1^*$ and $G\rho_2 \neq \rho_2^*$ ($\Delta\rho_1 \neq 0$ and $\Delta\rho_2 \neq 0$),

$$\underline{C}_n = \underline{B}_n .$$

(2) $G\rho_1 \neq \rho_1^*$ and $G\rho_2 = \rho_2^*$ ($\Delta\rho_1 \neq 0$ and $\Delta\rho_2 = 0$),

$$\underline{C}_n = \begin{bmatrix} \left.\frac{\partial \rho_{1,n+1}}{\partial P'}\right|_* & \left.\frac{\partial \rho_{1,n+1}}{\partial E'}\right|_* \\ \delta_2'^{-1}\left.\frac{\partial \rho_{2,n+1}}{\partial P'}\right|_* & \delta_2'^{-1}\left.\frac{\partial \rho_{2,n+1}}{\partial E'}\right|_* \end{bmatrix}.$$

(3) $G\rho_1 = \rho_1^*$ and $G\rho_2 \neq \rho_2^*$ ($\Delta\rho_1 = 0$ and $\Delta\rho_2 \neq 0$),

$$\underline{C}_n = \begin{bmatrix} \delta_1'^{-1}\left.\frac{\partial \rho_{1,n+1}}{\partial P'}\right|_* & \delta_1'^{-1}\left.\frac{\partial \rho_{1,n+1}}{\partial E'}\right|_* \\ \left.\frac{\partial \rho_{1,n+1}}{\partial P'}\right|_* & \left.\frac{\partial \rho_{1,n+1}}{\partial E'}\right|_* \end{bmatrix}.$$

(4) $G\rho_1 = \rho_1^*$ and $G\rho_2 = \rho_2^*$ ($\Delta\rho_1 = 0$ and $\Delta\rho_2 = 0$),

$$\underline{C}_n = \begin{bmatrix} \delta_1'^{-1}\left.\frac{\partial \rho_{1,n+1}}{\partial P'}\right|_* & \delta_1'^{-1}\left.\frac{\partial \rho_{1,n+1}}{\partial E'}\right|_* \\ \delta_2'^{-1}\left.\frac{\partial \rho_{2,n+1}}{\partial P'}\right|_* & \delta_2'^{-1}\left.\frac{\partial \rho_{2,n+1}}{\partial E'}\right|_* \end{bmatrix}.$$

Then, the eigenvalues $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ of the scaling matrix $\underline{\Gamma}_n$ are

$$\gamma^{(n)} = \frac{\mathrm{Tr}\underline{\Gamma}_n \pm [(\mathrm{Tr}\underline{\Gamma}_n)^2 - 4\,\mathrm{Det}(\underline{\Gamma}_n)]^{1/2}}{2} . \quad (3.15)$$

Note also that $\rho_1 = \rho_1(P)$ and $\rho_2(P,E)$ in the map (2.3) [see Eq. (2.9)]. Therefore we obtain the analytic formulas for $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ which depend on the values of $G\rho_1$ and $G\rho_2$ as follows.

(1) $G\rho_1 \neq \rho_1^*$ and $G\rho_2 \neq \rho_2^*$,

$$\gamma_1^{(n)} = \frac{d\rho_{1,n+1}/dP'\,\big|_*}{d\rho_{1,n}/dP\,\big|_*}, \quad \gamma_2^{(n)} = \frac{\partial\rho_{2,n+1}/\partial E'\,\big|_*}{\partial\rho_{2,n}/\partial E\,\big|_*} . \quad (3.16a)$$

(2) $G\rho_1 \neq \rho_1^*$ and $G\rho_2 = \rho_2^*$,

$$\gamma_1^{(n)} = \frac{d\rho_{1,n+1}/dP'|_*}{d\rho_{1,n}/dP|_*}, \quad \gamma_2^{(n)} = \frac{\partial\rho_{2,n+1}/\partial E'|_*}{\partial\rho_{2,n}/\partial E|_*}\delta_2'^{-1} . \tag{3.16b}$$

(3) $G\rho_1 = \rho_1^*$ and $G\rho_2 \neq \rho_2^*$,

$$\gamma_1^{(n)} = \frac{d\rho_{1,n+1}/dP'|_*}{d\rho_{1,n}/dP|_*}\delta_1'^{-1}, \quad \gamma_2^{(n)} = \frac{\partial\rho_{2,n+1}/\partial E'|_*}{\partial\rho_{2,n}/\partial E|_*} . \tag{3.16c}$$

(4) $G\rho_1 = \rho_1^*$ and $G\rho_2 = \rho_2^*$,

$$\gamma_1^{(n)} = \frac{d\rho_{1,n+1}/dP'|_*}{d\rho_{1,n}/dP|_*}\delta_1'^{-1} , \quad \gamma_2^{(n)} = \frac{\partial\rho_{2,n+1}/\partial E'|_*}{\partial\rho_{2,n}/\partial E|_*}\delta_2'^{-1} . \tag{3.16d}$$

As $n \to \infty$, $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ approach γ_1 and γ_2 which are the parameter-scaling factors,

$$\lim_{n\to\infty} \gamma_i^{(n)} = \gamma_i, \quad i = 1,2 . \tag{3.17}$$

By comparing the analytic formulas of $\delta_1^{(n)}$ and $\delta_2^{(n)}$ in Eq. (3.10) with those of $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ in Eq. (3.16), one can express the parameter-scaling factors γ_1 and γ_2 for any bifurcation path in terms of δ_1, δ_2, δ_1', and δ_2' as follows.

(1) $G\rho_1 \neq \rho_1^*$ and $G\rho_2 \neq \rho_2^*$ (regular path),

$$\gamma_1 = \delta_1 \text{ and } \gamma_2 = \delta_2 \tag{3.18a}$$

(2) $G\rho_1 \neq \rho_1^*$ and $G\rho_2 = \rho_2^*$ (the first type of special path),

$$\gamma_1 = \delta_1 \text{ and } \gamma_2 = \delta_2/\delta_2' . \tag{3.18b}$$

(3) $G\rho_1 = \rho_1^*$ and $G\rho_2 \neq \rho_2^*$ (the second type of special path),

$$\gamma_1 = \delta_1/\delta_1' \text{ and } \gamma_2 = \delta_2 . \tag{3.18c}$$

(4) $G\rho_1 = \rho_1^*$ and $G\rho_2 = \rho_2^*$ (the third type of special path):

$$\gamma_1 = \delta_1/\delta_1' \text{ and } \gamma_2 = \delta_2/\delta_2' . \tag{3.18d}$$

Therefore δ_1, δ_2, δ_1', and δ_2' are the "fundamental noncoordinate scaling factors," in the sense that the two-parameter scaling factors for any bifurcation path can be expressed in terms of them. Their values are shown in Table III. The values of δ_1 and δ_1' are the same for all bifurcation routes; moreover, these values are the same as the values of δ and δ' for period doubling in area-preserving maps (the value of δ' was found in Refs. 8 and 9). However, the values of δ_2 and δ_2' for the three kinds of bifurcation routes are different.

C. Orbital scaling factor

In this section we only review the orbital scaling behavior of the in-phase orbits. After making the linear transformation in Eq. (2.2), the old map (2.1) becomes the new map (2.3). Then, the first two coordinates X and Y in the new map (2.3) of the in-phase orbit with $U = V = 0$ are determined by the 2D Hénon map (2.4). Therefore X and Y scale with the 2D orbital scaling factors[2–9] $\alpha = -4.018\ldots$ and $\beta = 16.36\ldots$. Furthermore, according to the definition of the linear transformation (2.2), the coordinates x and y (or u and v) of the in-phase orbit also scale with the same 2D orbital scaling factors α and β.

IV. PERIOD-2 SCALING PATTERN

In this section the results of the period-2 scaling pattern in the period-2 bifurcation routes are given. There is only one kind of period-2 bifurcation route whose address is $[d,(L,L,U,U,)^\infty]$. To perform a two-parameter search,[13] we consider the case that the values of (F,G) are (1,2), (2,4), (1,3), and (2,3) and follow the orbit whose level n is up to 19. The parameter-scaling behavior is independent of the values of F and G within numerical accuracy. This is expected as we are considering a codimension-two problem. We numerically obtain the four fundamental noncoordinate-scaling factors δ_1, δ_2, δ_1', and δ_2'. Furthermore, we analytically obtain the value of δ_2 and the critical stability index ρ_2^* by an extremely simply renormalization method.[21]

We first explain the meaning of period-2 scaling pattern. The period-doubling sequence repeats itself asymptotically from one bifurcation to every other one in a period-2 bifurcation route, whereas the period-doubling sequence repeats itself from one bifurcation to the next one in a period-1 bifurcation route. As an example, let us consider the period-2 route whose address is $[(L,L,U,U,)^\infty]$ when $F=1$ and $G=2$. The accumulation point (P^*,E^*) in this period-2 route is

$$P^* = -1.266\,311\,276\,922\,099\,113\,716\,3675 , \quad E^* = -1.157\,360\,600\,056\,928\,496\,228\,813 . \tag{4.1}$$

At the accumulation point, the stability indices $\rho_{1n}(P^*)$ and $\rho_{2,n+1}(P^*,E^*)$ are shown in Table IV. As shown in Table IV, $(\rho_{1,n},\rho_{2,n})$ exhibits a period-2 pattern and converges to the critical stability index (ρ_1^*,ρ_2^*),

$$(\rho_1^*,\rho_2^*) = \lim_{n\to\infty}(\rho_{1,n},\rho_{2,n}) = \begin{cases} (\rho_1^*,\rho_{2,e}^*) & \text{for } n \text{ even} \\ (\rho_1^*,\rho_{2,o}^*) & \text{for } n \text{ odd.} \end{cases} \tag{4.2}$$

Here, we denote the critical stability index ρ_2^* by $\rho_{2,e}^*$ for n even and by $\rho_{2,o}^*$ for n odd. The values of ρ_1^*, $\rho_{2,e}^*$, and $\rho_{2,o}^*$ are shown in Table V. This implies that there exists a period-2 map of the renormalization transformation.

Second, we obtain the four fundamental noncoordinate scaling factors δ_1, δ_2, δ_1', and δ_2'. At the accumulation point (P^*,E^*) in a period-2 bifurcation route, $\rho_{1,n}(P^*)$ and $\rho_{2,n}(P^*,E^*)$ converge to the critical stability indices ρ_1^* and ρ_2^* with rates δ_1' and δ_2', respectively.

(1) n even,

TABLE IV. The stability indices $\rho_{1,n}(P^*)$ and $\rho_{2,n}(P^*,E^*)$ of a periodic orbit of level n in a period-2 bifurcation route whose address is $[(L,L,U,U,)^\infty]$ for $F=1$ and $G=2$.

n	$\rho_{1,n}(P^*)$	$\rho_{2,n}(P^*,E^*)$
5	−2.543 510 411 332	−1.619 112 861 583
6	−2.543 510 172 372	0.617 979 560 460
7	−2.543 510 200 259	−1.618 035 334 750
8	−2.543 510 197 008	0.618 034 639 559
9	−2.543 510 197 387	−1.618 034 073 156
10	−2.543 510 197 343	0.618 033 975 676
11	−2.543 510 197 348	−1.618 033 987 845
12	−2.543 510 197 347	0.618 033 988 971
13	−2.543 510 197 347	−1.618 033 988 769
14	−2.543 510 197 349	0.618 033 988 746
15	−2.543 510 197 305	−1.618 033 988 750
16	−2.543 510 197 478	0.618 033 988 750

$$\rho_{1,n}-\rho_1^* \sim \delta_1'^n , \quad \rho_{2,n}-\rho_{2,e}^* \sim \delta_2'^n ; \tag{4.3a}$$

(2) n odd,

$$\rho_{1,n}-\rho_1^* \sim \delta_1'^n , \quad \rho_{2,n}-\rho_{2,o}^* \sim \delta_2'^n . \tag{4.3b}$$

The values of δ_1' and δ_2' for n even are the same as those for n odd, and they are shown in Table VI. The value of δ_1' for the period-2 scaling pattern corresponds to the square of the value of δ_1' for the period-1 scaling pattern since $\rho_{1,n}(P^*)$ itself exhibits a period-1 pattern as shown in Table IV. Since $|\delta_2'|>|\delta_1'|$, δ_2' is the "essential" convergence rate.[9] The divergence rates δ_1 and δ_2 from the period-2 map can be obtained by the scaling-matrix method.[19] For the period-2 scaling pattern, the 2×2 scaling matrix $\underline{\Gamma}_n$ is defined as follows:

$$\begin{bmatrix} P_n-P_{n-2} \\ E_n-E_{n-2} \end{bmatrix} = \underline{\Gamma}_n \begin{bmatrix} P_{n+2}-P_n \\ E_{n+2}-E_n \end{bmatrix} . \tag{4.4}$$

Then, $\underline{\Gamma}_n$ approaches a constant matrix $\underline{\Gamma}$ as $n\to\infty$. The eigenvalues of the constant matrix $\underline{\Gamma}$, γ_1 and γ_2, are the parameter-scaling factors. As shown in III B, the scaling factors γ_1 and γ_2 for the regular paths are just the divergence rates δ_1 and δ_2 [see Eq. (3.18a)]. The values of δ_1 and δ_2 for n even are the same as those for n odd, and they are shown in Table VI. Note that the value of δ_1 for the period-2 scaling pattern corresponds to the square of the value of δ_1 for the period-1 scaling pattern. There exist three kinds of special bifurcation paths. However, as shown in Eqs. (3.18), the scaling factors γ_1 and γ_2 for any special bifurcation path are some combination of the four fundamental noncoordinate scaling factors.

Third, we obtain analytically the values of δ_2 and ρ_2^* by an extremely simple renormalization method.[21] The stability index ρ_2 of a periodic orbit of level n is

$$\rho_2=\lambda_2+\lambda_2^{-1} , \tag{4.5}$$

where λ_2 is a multiplier of the periodic orbit. Since we are considering the period-2 scaling pattern, we next obtain the stability index ρ_2' of a periodic orbit of level $n+2$, where

$$\rho_2'=\lambda_2^4+\lambda_2^{-4} . \tag{4.6}$$

By expressing ρ_2' in terms of ρ_2, we obtain a recurrence relation for ρ_2

$$\rho_2'=\rho_2^4-4\rho_2^2+2 . \tag{4.7}$$

The critical stability index ρ_2^* is a fixed point of the recurrence relation (4.7). That is, ρ_2 is a root of the following equation:

$$\rho_2^{*4}-4\rho_2^{*2}-\rho_2^*+2=0 , \tag{4.8}$$

whose roots are

$$\rho_2^*=2,\ -1,\ (-1+\sqrt{5})/2, \text{ and } (-1-\sqrt{5})/2 . \tag{4.9}$$

The divergence rate δ_2 can be determined by the equation

$$\delta_2=\left.\frac{\partial\rho_2'}{\partial\rho_2}\right|_{\rho_2^*} . \tag{4.10}$$

The values of δ_2 are as follows:

$$\delta_2=\begin{cases} 16 \text{ for } \rho_2^*=2 , \\ 4 \text{ for } \rho_2^*=-1 , \\ -4 \text{ for } \rho_2^*=(-1+\sqrt{5})/2 \text{ or } (-1-\sqrt{5})/2 . \end{cases} \tag{4.11}$$

The values of ρ_2^* in the S and A routes, which are period-1 bifurcation routes, are equal to 2 and −1, respectively. Moreover, the values of δ_2 in the S and A routes are 4 and −2, respectively. Thus, if one compares the period-doubling pattern of level n with that of level $n+2$, then the values of δ_2 become 16 and 4. Therefore the two values of ρ_2^* ($\rho_2^*=2$ and −1) in Eq. (4.9) must be excluded, as these are just the values of ρ_2^* in the period-1 bifurcation routes (S and A routes). Then, there remain only two values of ρ_2^* and one value of δ_2:

$$\delta_2=-4 \quad \text{for } \rho_2^*=(-1+\sqrt{5})/2 \text{ or } (-1-\sqrt{5})/2 . \tag{4.12}$$

These values of ρ_2^* and δ_2 in Eq. (4.12) agree very well

TABLE V. The critical stability indices ρ_1^*, $\rho_{2,e}^*$, and $\rho_{2,o}^*$ in a period-2 bifurcation route.

Route	ρ_1^*	$\rho_{2,e}^*$	$\rho_{2,o}^*$
Period 2	−2.543 510 197 347	0.618 033 988 750	−1.618 033 988 750

TABLE VI. The four fundamental noncoordinate scaling factors δ_1, δ_2, δ_1' and δ_2' in a period-2 bifurcation route.

Route	δ_1	δ_2	δ_1'	δ_2'
Period 2	76.06	−4.000	0.013 60	−0.017 59

with the numerical values (see Tables V and VI).

Finally, we mention briefly the orbital scaling behavior of the in-phase orbit in the period-2 bifurcation route. As reviewed in Sec. III C, the coordinates x and y (or u and v) scale with the same 2D orbital scaling factors $\alpha=-4.018\ldots$ and $\beta=-16.36\ldots$. Therefore, in a period-2 bifurcation route, although the parameter sequence exhibits a period-2 scaling pattern, the orbital sequence exhibits a period-1 scaling pattern.

V. SUMMARY

By generalizing the pattern bifurcation route and bifurcation path, we have studied the period-M ($M=1$ and 2) scaling pattern of period doubling in a symmetric 4D volume-preserving map. The parameter-scaling factors γ_1 and γ_2 depend on the bifurcation path. However, it was shown that the two parameter-scaling factors can be expressed in terms of the four "fundamental noncoordinate scaling factors" δ_1, δ_2, δ_1', and δ_2'. Therefore each bifurcation route is characterized by its own four fundamental noncoordinate scaling factors. For the period-1 scaling pattern, there are three kinds of period-1 bifurcation routes (S, A, and E routes). There are three kinds of special paths for the S and A routes, and two kinds of special paths for the E route. Although the period-1 scaling pattern was previously studied,[13–15] the four fundamental noncoordinate scaling factors were not found, and only one special path for each bifurcation route was found. Furthermore, we have found a new scaling pattern in the period-2 bifurcation route, called the period-2 scaling pattern. The period-doubling sequence repeats itself from one bifurcation to every other one in a period-2 bifurcation route. We have obtained the four fundamental noncoordinate-scaling factors for the period-2 scaling pattern by a numerical method. Moreover, we have obtained the values of δ_2 and ρ_2^* by a simple renormalization method. The values of δ_1 and δ_1' in any period-M bifurcation route are the same as those in area-preserving maps. However, the values of δ_2 and δ_2' depend on the bifurcation route. Moreover, since $|\delta_2'|\geq|\delta_1'|$ in any period-M bifurcation route (equality holds only for the E route), δ_2' is the "essential" convergence rate of a critical map in the scaling coordinate.

We introduced the route sequence of a bifurcation route in Sec. II D. The route sequence of a period-M bifurcation route exhibits eventually a period-M behavior. From this fact, we conjecture that in a bifurcation route whose route sequence exhibits eventually a period-M (M any positive integer) behavior, the period-doubling sequence exhibits asymptotically a period-M scaling pattern. In a bifurcation route whose route-sequence is random, it exhibits a "chaotic" scaling pattern. Note that for an invariant circle in area-preserving maps, if the continued-fraction representation of its rotation number has a "period-M" tail, then it exhibits a period-M scaling pattern; otherwise, it exhibits a "chaotic" scaling pattern.[9,22–25] Therefore we conjecture that the route sequence plays the same role as the continued-fraction representation. Thus all the scaling patterns of period doublings could be classified by the route sequence.

Note that we have studied only symmetric 4D volume-preserving maps. The period-1 scaling patterns in nonsymmetric 4D volume-preserving maps is the same as those in symmetric 4D volume-preserving maps.[15,21] We conjecture that the period-2 scaling pattern in nonsymmetric 4D volume-preserving maps is also the same as those in symmetric 4D volume-preserving maps. This is because the simple renormalization analysis for the period-2 scaling pattern in Sec. IV can be applied to any (symmetric or nonsymmetric) 4D volume-preserving maps and gives the same results. Therefore it would be desirable to check the analytic results of the simple renormalization analysis for the period-2 scaling pattern by studying numerically the scaling pattern in a nonsymmetric 4D volume-preserving map.

ACKNOWLEDGMENTS

We would like to acknowledge useful discussions with Dr. J. M. Mao. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG05-87ER40374, the Houston Area Research Consortium Supercomputer Grant No. 88-017, and the Non-Directed Research Fund, Korea Research Fund, 1989.

[1]M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979).
[2]G. Benettin, C. Cercignani, L. Galgani, and A. Giorgilli, Lett. Nuovo Cimento **28**, 1 (1980); **29**, 163 (1980).
[3]P. Collet, J. P. Eckmann, and H. Koch, Physica D **3**, 457 (1981).
[4]J. Greene, R. S. MacKay, F. Vivaldi, and M. J. Feigenbaum, Physica **D3**, 468 (1981).
[5]T. C. Bountis, Physica **D3**, 577 (1981).
[6]T. Janssen and J. A. Tjon, Phys. Lett. **87A**, 139 (1982).
[7]R. H. G. Helleman, in *Long-Time Prediction in Dynamics*, edited by W. Horton, L. Reichl, and V. Szebehely (Wiley, New York, 1982), pp. 95–126.
[8]M. Widom and L. P. Kadanoff, Physica D **5**, 287 (1982).
[9]R. S. MacKay, Ph.D. thesis, Princeton University, 1982.
[10]P. Collet, J. P. Eckmann, and H. Koch, J. Stat. Phys. **25**, 1 (1981).

[11]B. Hu and J. M Mao, Phys. Rev. A **29**, 1564 (1984).
[12]J. Janssen and T. A. Tjon, J. Phys. A **16**, 673 (1983); **16**, 697 (1983).
[13]J. M. Mao, I. Satija, and B. Hu, Phys. Rev. A **32**, 1927 (1985); **34**, 4325 (1986).
[14]J. M. Mao and R. H. G. Helleman, Phys. Rev. A **35**, 1847 (1987).
[15]J. M. Mao and J. Greene, Phys. Rev. A **35**, 3911 (1987).
[16]V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978).
[17]R. Broucke, AIAA J. **7**, 1003 (1969).
[18]J. E. Howard and R. S. MacKay, J. Math. Phys. **28**, 1036 (1987).
[19]J. Guckenheimer, B. Hu, and J. Rudnick (unpublished); B. Hu and J. M. Mao, *Directions in Chaos,* edited by B. L. Hao (World Scientific, Singapore, 1988), Vol. 1, p. 206.
[20]B. Derrida, A. Gervois, and Y. Pomeau, J. Phys. A **12**, 269 (1979); B. Derrida and Y. Pomeau, Phys. Lett. **80A**, 217 (1980); B. Hu and J. M. Mao, Phys. Rev. A **25**, 1196 (1982); **25**, 3259 (1982); Phys. Lett. A **108**, 305 (1985).
[21]J. M. Mao and R. H. G. Helleman, Phys. Rev. A **37**, 3475 (1988).
[22]S. J. Shenker and L. P. Kadanoff, J. Stat. Phys. **27**, 631 (1982).
[23]R. S. MacKay, Physica **D7**, 283 (1983).
[24]J. M. Greene, R. S. MacKay, and J. Stark, Physica D **21**, 267 (1986).