Critical phenomena of invariant circles

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Some novel critical phenomena are discovered in a class of nonanalytic twist maps. It is found that the degree of inflection z plays a role reminiscent of that of dimensionality in phase transitions with $z=2$ and 3 corresponding to the lower and upper critical dimensions, respectively. Moreover, recurrence of invariant circles has also been observed. An "inverse residue criterion," complementary to the "residue criterion" for the determination of the disappearance point, is introduced to determine the reappearance point of invariant circles.

The Kolmogorov-Arnold-Moser (KAM) theorem was a landmark in the history of mechanics—in a sense, it ushered in the modern era of mechanics. Its impact, however, far exceeds the confines of mechanics. The relevance and importance of the KAM theorem have been recognized by a rapidly increasing number of other fields, some of which are not even remotely related to mechanics.

As the original proof of the KAM theorem goes, its validity is restricted to an extremely small perturbation. An important contribution was made by Greene, whose work[1] can be viewed as an extension of the KAM theorem from the weak- to the strong-coupling regime. In particular, his "residue criterion" provides a precise method to determine the transition point at which the last invariant circle disappears.

Another major advancement was attained under the influence of the theory of phase transitions. The method of the renormalization group, which led to an epoch-making breakthrough in the theory of phase transitions in the last decade, has also led to a remarkable breakthrough in the theory of stochastic transitions in the present decade. Universal scaling behavior[2,3] near the transition point has been discovered and understood. More recently, a theory of transport[4,5] above the transition point has also been formulated. It thus seems we have achieved a fairly complete understanding of the life story of an invariant circle.

However, one outstanding defect in these otherwise perfect studies is that most of the results obtained so far are based on the standard map, which might be aptly called the "standard model" of stochastic transitions. It is therefore not clear how general the conclusions are beyond the standard model. Greene's criterion, albeit it has worked amazingly well, is still in want of a proof.

To test the generality of the standard model, we propose to study a class of modified twist maps:

$$T:\begin{cases} r_{i+1}=r_i-kg(\theta_i) \\ \theta_{i+1}=\theta_i+r_{i+1} \pmod 1 \end{cases} \tag{1}$$

where $\theta_i\in[-\frac{1}{2},\frac{1}{2})$ and $r_i\in[0,1]$. The function $g(\theta)=\theta(1-|2\theta|^{z-1})$ is so chosen that it has a variable degree of inflection z at the inflection point $\theta=0$. $g(\theta)$ is nonanalytic (C^1). The motivation for employing such a function is that in the circle map, which can be regarded as the dissipative limit of the standard map, the degree of inflection serves as a universality criterion. The sine function in the circle map possesses a cubic inflection point ($z=3$). To generalize it to an arbitrary degree of inflection, $g(\theta)$ was invented.[6,7] Although the substitution of this function in the circle map produced no surprising results, many unexpected novel features have been discovered in the conservative case.

We have studied the behavior of the "golden-mean" invariant circle as we vary the degree of inflection. In the following we report some of our main findings.

(1) $k<0$ or $z\leq 2$. We found that, for $k\neq 0$, the residue tends to infinity and the action difference[8] tends to a nonzero constant as the period tends to infinity. It suggests that there are no invariant circles when $k\neq 0$. As in the sawtooth map, $k_D=0$ is therefore the trivial critical point for the breakup of the invariant circle. This is similar to a zero-temperature phase transition.

(2) $2<z\leq 3$. In this case $k_D\neq 0$; however, no reappearance of invariant circles has been observed. k_D tends to zero monotonically as z decreases from 3 to 2 (see Table I).

(3) $z>3$. Reappearance of invariant circles has been observed.[9–11] We found that there is more than one value of k which satisfies Green's criterion. For example, for $z=4$, two disappearance points $k_D^{(1)}=1.412\,935\,3$ and $k_D^{(2)}=1.426\,155\,7$ have been found. If an invariant circle disappears at two points, there must be a point k_R,

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TABLE I. Disappearance points k_D and the scaling exponents for $2 \leq z \leq 3$. The superscript (3) denotes period-3 scaling. The subscripts s and n denote the scaling on the dominant and the nondominant symmetry line, respectively.

z	k_D	$x_s^{(3)}$	$y_s^{(3)}$	$x_s^{(3)}+y_s^{(3)}$	$x_n^{(3)}$	$y_n^{(3)}$	$x_n^{(3)}+y_n^{(3)}$	$\delta^{(3)}$
2.0	0	1	2	3	1	2	3	
2.1	0.219	0.965	2.036	3.001	1.005	1.996	3.001	1.27
2.2	0.391	0.935	2.067	3.002	1.013	1.989	3.002	1.42
2.3	0.5375	0.907	2.098	3.005	1.021	1.984	3.005	1.55
2.4	0.6617	0.881	2.127	3.008	1.030	1.978	3.008	1.75
2.5	0.768 28	0.858	2.155	3.013	1.038	1.975	3.013	1.97
2.6	0.860 37	0.835	2.181	3.016	1.046	1.970	3.016	2.23
2.7	0.940 34	0.810	2.211	3.021	1.055	1.967	3.022	2.53
2.8	1.010 114	0.788	2.239	3.027	1.063	1.964	3.027	2.88
2.9	1.071 375	0.763	2.271	3.034	1.073	1.961	3.034	3.36
3.0	1.125 454	0.721	2.330	3.051	1.092	1.959	3.051	4.25

$k_D^{(1)} < k_R < k_D^{(2)}$, at which it reappears. To determine k_R, we introduce an "inverse version of residue criterion" for the reappearance of an invariant circle.

An invariant circle with winding number ω that has disappeared at k_D will reappear at k_R if

$$\lim_{i\to\infty} R_i^{\pm}(k) = \begin{cases} \pm\infty, & k < k_R \\ R^{\pm}, & k = k_R \\ 0^{\pm}, & k > k_R \ . \end{cases} \quad (2)$$

$R_i^{\pm}(k)$ are the residues of the minimax (+) and minimizing (−) orbits with winding numbers $\omega_i = P_i/Q_i$ at a given value of k. $R^{\pm}$ are two constants; $|R^{\pm}| < 1$. This "inverse residue criterion" enables us to make a precise determination of the reappearance point. It is complementary to the "residue criterion" for the determination of the disappearance point. For example, for $z=4$, $k_R^{(1)} = 1.421\,734\,15$; and for $z=6$, $k_R^{(1)} = 1.299\,465\,40$, $k_R^{(2)} = 1.452\,963\,40$. The superscript i in $k_{D,R}^{(i)}$ refers to the ith time the invariant circle disappears (D) or reappears (R). We have also computed the action difference and found $\Delta W \to 0$ if $k < k_D^{(1)}$ or $k_R^{(1)} < k < k_D^{(2)}$; and $\Delta W \to \text{const} > 0$ if $k_D^{(1)} < k < k_R^{(1)}$ or $k > k_D^{(2)}$. Figure 1 shows a typical case. Figure 2 shows the evolution of the phase portrait from disappearance to reappearance of the invariant circle for the case $z=6$. In Fig. 2(a), $k < k_D^{(1)}$, the chaotic regions near the period-1 resonance and the period-2 resonance are separated by the invariant circle. In Fig. 2(b), $k_D^{(1)} < k < k_R^{(1)}$, the invariant circle has disappeared and the chaotic regions become connected. In Fig. 2(c), $k_R^{(1)} < k < k_D^{(2)}$, the invariant circle has reappeared and the chaotic regions become separated again. These results together with the scaling behavior to be discussed later suggest that $k_R^{(i)}$ are indeed the points at which the invariant circle reappears. Moreover, an invariant circle can recur more than once. For example, for $z=6$, we have observed that the invariant circle has recurred at least twice. The residues are no longer monotonic functions of k. They tend to infinity right after the invariant circle has disappeared, and become finite again as it reappears. Since we cannot ascertain the existence of a "final" disappearance, the invariant circle can conceivably recur infinitely many times. We have also observed an exchange of stability before the first breakup of the invariant circle. Figure 3 shows the variation of the residues with k. It is evident here that stability exchange does not necessarily entail reappearance. When z is a fraction, the dependence of the residues on k is quite complicated. A "bifurcation" of the regions in which the invariant circle exists has also been observed as z varies.

The scaling behavior of the invariant circle at the critical points of disappearance and reappearance have also been studied. We first summarize the definitions of various scaling exponents. For a pair of minimizing and minimax orbits with winding number P_i/Q_i, the distance between two neighboring orbit points is $d_i = |\theta_i^{\max} - \theta_i^{\min}|$, where the superscripts "max" and "min" denote the

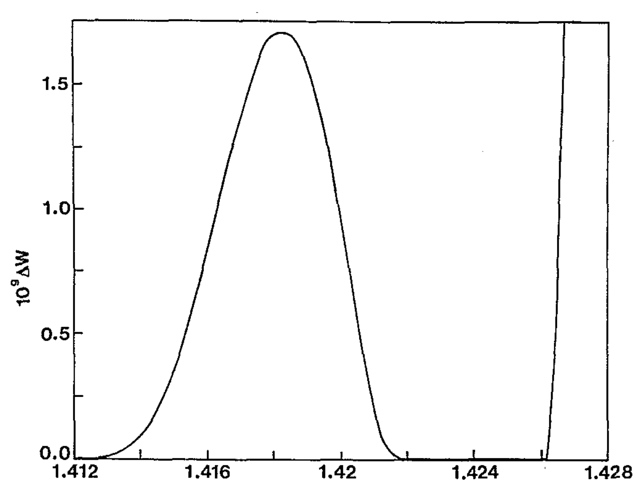

FIG. 1. The action difference ΔW as a function of k for $z=4$ and $(Q_i, P_i) = (1597, 987)$.

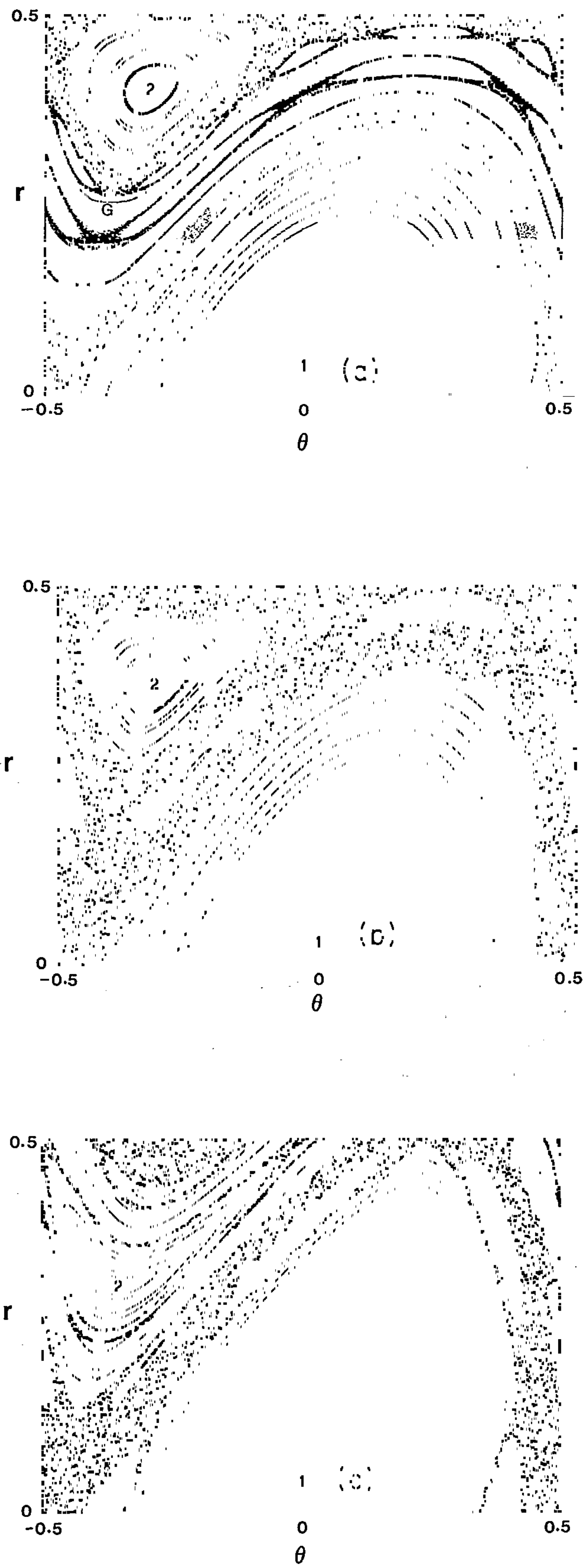

FIG. 2. Phase portraits from disappearance to reappearance of the invariant circle for the case $z=6$. 1, 2, and G indicate, respectively, the period-1 resonance, period-2 resonance, and the "golden-mean" invariant circle. (a) $k=0.704<k_D^{(1)}$; (b) $k_D^{(1)}<k=0.9<k_R^{(1)}$; (c) $k_R^{(1)}<k=1.35<k_D^{(2)}$.

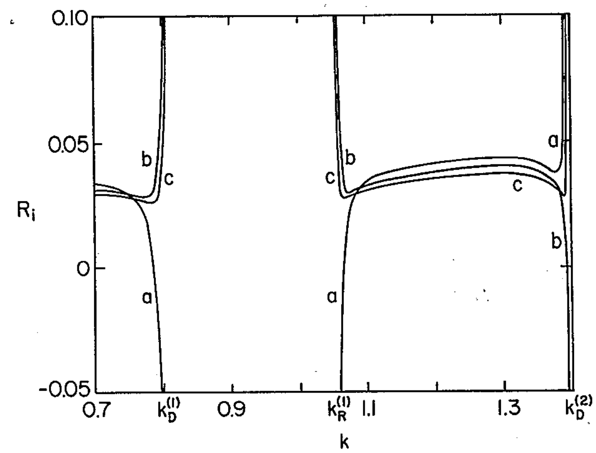

FIG. 3. The residues R_i of the initially elliptic orbits for $z=5$. Curves a, $(Q_i,P_i)=(233,144)$; curves b, (377,233); curves c (610,377). The residues of the initially hyperbolic orbits are symmetrically located, and are not shown. Stability exchange occurs before the first breakup of the invariant circle for curves a, but not for curves b and c.

minimax and minimizing orbits, respectively. Period-1 scaling is defined by

$$\frac{d_i}{d_{i+1}} \sim \left[\frac{Q_i}{Q_{i+1}}\right]^{-x^{(1)}}, \qquad \frac{r_i-r_{i-1}}{r_{i+1}-r_i} \sim \left[\frac{Q_i}{Q_{i+1}}\right]^{-y^{(1)}}; \tag{3}$$

and period-3 scaling is defined by

$$\frac{d_i}{d_{i+3}} \sim \left[\frac{Q_i}{Q_{i+3}}\right]^{-x^{(3)}}, \qquad \frac{r_i-r_{i-3}}{r_{i+3}-r_i} \sim \left[\frac{Q_i}{Q_{i+3}}\right]^{-y^{(3)}}, \tag{4}$$

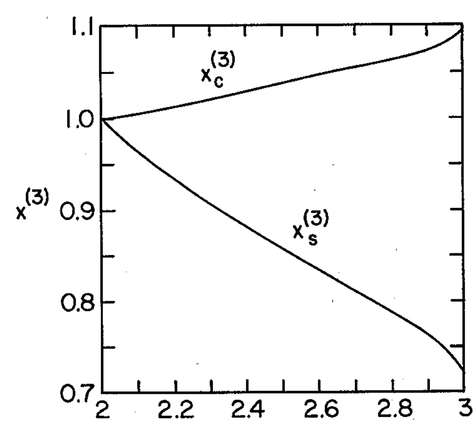

FIG. 4. The scaling exponents on the dominant symmetry line $x_s^{(3)}$ and the nondominant symmetry line $x_n^{(3)}$ as a function of z for $2\le z\le 3$.

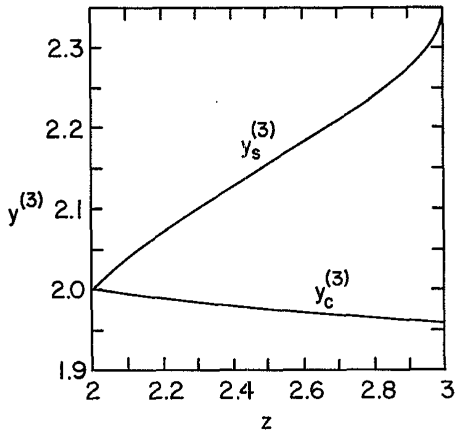

FIG. 5. The scaling exponents on the dominant symmetry line $y_s^{(3)}$ and the nondominant symmetry line $y_n^{(3)}$ as a function of z for $2 \leq z \leq 3$.

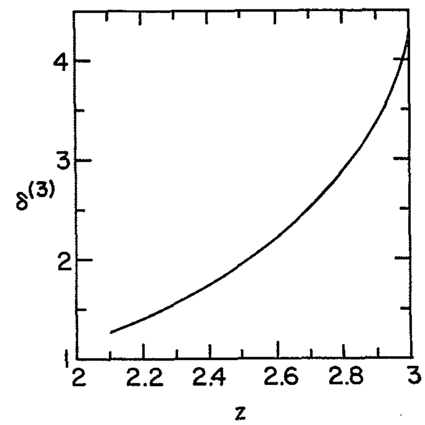

FIG. 6. The parameter scaling exponent $\delta^{(3)}$ as a function of z for $2 \leq z \leq 3$.

where $x^{(i)}$ and $y^{(i)}$ with $i=1$ or 3 are the scaling exponents which are related to the usual α and β by $\alpha=\omega^{-x}$ and $\beta=\omega^{-y}$ with $\omega=(\sqrt{5}-1)/2$. Due to its better convergence, period-3 scaling is used. For convenience, the exponents on the dominant symmetry line are denoted by $x_s^{(i)}$ and $y_s^{(i)}$; and the exponents on the symmetry line l ($l=a,b,c,d$) are denoted by $x_l^{(i)}$ and $y_l^{(i)}$. For the standard map[2] it was found that $x_s^{(1)}=0.721$, $y_s^{(1)}=2.329$, $x_a^{(3)}=1.093$, $y_a^{(3)}=2.329$, and $x_s^{(1)}+y_s^{(1)}=x_a^{(3)}+y_a^{(3)}=3.05$ at the critical point of the breakup of the invariant circle. The convergence rate of the parameter gives another scaling exponent. Again period-3 scaling is used:

$$\delta^{(3)}=\frac{k_{n+1}-k_n}{k_{(n+3)+1}-k_{n+3}}, \quad (5)$$

which is related to the usual δ by $\delta^{(3)}=\delta^3$. For the standard map $\delta^{(3)}=(1.628)^3$.

For $k<0$ or $z \leq 2$ the critical point is $k_D=0$, and the system is integrable. The scaling exponents can be calculated analytically: $x=1$ and $y=2$. This is just the scaling behavior of a linear system. For $2<z<3$, we found that the scaling exponents at the critical point k_D vary with z (see Table I and Figs. 4–6). For $z=3$ the scaling behavior is the same as that of the standard map. Therefore the scaling behavior changes smoothly from that of a linear system to that of the standard map as z varies from 2 to 3. The sum of the exponents, $x+y$, which is a more useful quantity in the study of transport, shows a slightly increasing trend. However, the increase is too small to exclude the possibility that it is in fact a constant. Since δ decreases with z, higher-period orbits are needed to compute the exponents. The orbits used here are up to

TABLE II. Disappearance (k_D) and reappearance (k_R) points and the scaling exponents for $z>3$.

z	k	$x_s^{(3)}$	$y_s^{(3)}$	$x_s^{(3)}+y_s^{(3)}$	$x_n^{(3)}$	$y_n^{(3)}$	$x_n^{(3)}+y_n^{(3)}$
3.8	$k_D^{(1)}=1.382\,534\,50$	0.7284	2.3261	3.0545	1.0990	1.9448	3.0438
	$k_R^{(1)}=1.387\,603\,67$	0.7226	2.3295	3.0521	1.1018	1.9512	3.0530
4	$k_D^{(1)}=1.412\,935\,30$	0.7219	2.3287	3.0506	1.1045	1.9419	3.0464
	$k_R^{(1)}=1.421\,734\,15$	0.7223	2.3281	3.0504	1.1015	1.9466	3.0481
	$k_D^{(2)}=1.426\,155\,70$	0.7203	2.3329	3.0533	1.0986	1.9425	3.0412
5	$k_D^{(1)}=0.809\,930\,00$	0.7234	2.3281	3.0515	1.1030	1.9432	3.0462
	$k_R^{(1)}=1.052\,873\,50$	0.7216	2.3288	3.0504	1.1040	1.9432	3.0472
	$k_D^{(2)}=1.396\,474\,20$	0.7221	2.3285	3.0505	1.1044	1.9422	3.0466
6	$k_D^{(1)}=0.704\,000\,46$	0.7218	2.3289	3.0507	1.1046	1.9421	3.0467
	$k_R^{(1)}=1.299\,465\,40$	0.7227	2.3284	3.0511	1.1037	1.9432	3.0470
	$k_D^{(2)}=1.437\,318\,67$	0.7228	2.3304	3.0532	1.1121	1.9327	3.0448
	$k_R^{(2)}=1.452\,963\,40$	0.7229	2.3303	3.0533	1.1121	1.9326	3.0447
	$k_D^{(3)}=1.512\,575\,48$	0.7221	2.3293	3.0513	1.1024	1.9454	3.0478
11	$k_D^{(1)}=0.278\,305\,53$	0.7211	2.3303	3.0515	1.1021	1.9462	3.0483

$Q_i = 75\,025$. As z is close to 2, the convergence of δ_n towards δ is slowed down. It becomes very hard to evaluate the exponents for z near 2. When $z \geq 3$, it was found that the exponents at the disappearance and reappearance points are equal and are the same as those of the standard map. They are also independent of z (see Table II). In this sense the degree of inflection z plays a role quite similar to that of dimensionality in phase transitions with $z = 2$ and 3 corresponding to the lower and upper critical dimensions, respectively.

The many novel features[12,13] observed in this work suggest that the behavior of invariant circles may in fact be much more complicated than we have been accustomed to think. It is thus worthwhile to conduct a more thorough study of the KAM theorem as well as the Greene criterion beyond the standard model. The importance of such a study lies not only in its intrinsic value, but also in its potential applicability to many diverse fields.

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