Extension of “Renormalization of period doubling in symmetric four-dimensional volume-preserving maps”

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We numerically reexamine the scaling behavior of period doublings in four-dimensional volume-preserving maps in order to resolve a discrepancy between numerical results on scaling of the coupling parameter and the approximate renormalization results reported by Mao and Greene [Phys. Rev. A 35, 3911 (1987)]. In order to see the fine structure of period doublings, we extend the simple one-term scaling law to a two-term scaling law. Thus we find a new scaling factor associated with coupling and confirm the approximate renormalization results.

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Universal scaling behavior of period doubling has been found in area-preserving maps [1–7]. As a nonlinearity parameter is varied, an initially stable periodic orbit may lose its stability and give rise to the birth of a stable period-doubled orbit. An infinite sequence of such bifurcations accumulates at a finite parameter value and exhibits a universal limiting behavior. However, these limiting scaling behaviors are different from those for the one-dimensional dissipative case [8].

An interesting question is whether the scaling results of area-preserving maps carry over higher-dimensional volume-preserving maps. Thus period doubling in four-dimensional (4D) volume-preserving maps has been much studied in recent years [7,9–13]. It has been found in Refs. [11–13] that the critical scaling behaviors of period doublings for two symmetrically coupled area-preserving maps are much richer than those for the uncoupled area-preserving case. There exist an infinite number of critical points in the space of the nonlinearity and coupling parameters. It has been numerically found in [11,12] that the critical behaviors at those critical points are characterized by two scaling factors, $\delta_1$ and $\delta_2$. The value of $\delta_1$ associated with scaling of the nonlinearity parameter is always the same as that of the scaling factor $\delta$ (= 8.721 … ) for the area-preserving maps. However, the values of $\delta_2$ associated with scaling of the coupling parameter vary depending on the type of bifurcation routes to the critical points.

The numerical results [11,12] agree well with the approximate analytic renormalization results obtained by Mao and Greene [13], except for the zero-coupling case in which the two area-preserving maps become uncoupled. Using an approximate renormalization method including truncation, they found three relevant eigenvalues, $\delta_1 = 8.9474$, $\delta_2 = -4.4510$, and $\delta_3 = 1.8762$ for the zero-coupling case [14]. However, they believed that the third one, $\delta_3$, is an artifact of the truncation, because only two relevant eigenvalues $\delta_1$ and $\delta_2$ could be identified with the scaling factors numerically found.

In this Brief Report we numerically study the critical behavior at the zero-coupling point in two symmetrically coupled area-preserving maps and resolve the discrepancy between the numerical results on the scaling of the coupling parameter and the approximate renormalization results for the zero-coupling case. In order to see the fine structure of period doublings, we extend the simple one-term scaling law to a two-term scaling law. Thus we find a new scaling factor $\delta_3 = 1.8505$ … associated with coupling, in addition to the previously known coupling scaling factor $\delta_2 = -4.4038$ … . The numerical values of $\delta_2$ and $\delta_3$ are close to the renormalization results of the relevant coupling eigenvalues $\delta_2$ and $\delta_3$. Consequently, the fixed map governing the critical behavior at the zero-coupling point has two relevant coupling eigenvalues $\delta_2$ and $\delta_3$ associated with coupling perturbations, unlike the cases of other critical points.

Consider a 4D volume-preserving map $T$ consisting of two symmetrically coupled area-preserving Hénon maps [11,12],

$$
\begin{align*}
T: \quad & x_1(t+1) = -y_1(t) + f(x_1(t)) + g(x_1(t), x_2(t)) \\
& y_1(t+1) = x_1(t) \\
& x_2(t+1) = -y_2(t) + f(x_2(t)) + g(x_2(t), x_1(t)) \\
& y_2(t+1) = x_2(t),
\end{align*}
$$

(1)

where $t$ denotes a discrete time, $f$ is the nonlinear function of the uncoupled Hénon quadratic map [15], i.e.,

$$
f(x) = 1 - ax^2,
$$

(2)

and $g(x_1, x_2)$ is a coupling function obeying a condition

$$
g(x, x) = 0 \text{ for any } x.
$$

(3)

The two-coupled map (1) is called a symmetric map [11,12] because it is invariant under an exchange of coordinates such that $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$. The set of all points, which are invariant under the exchange of coordinates, forms a symmetry plane on which $x_1 = x_2$ and $y_1 = y_2$. An orbit is called an in-phase orbit if it lies on the symmetry plane, i.e., it satisfies

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\( x_1(t) = x_2(t) \equiv x(t), \quad y_1(t) = y_2(t) \equiv y(t) \) for all \( t \). (4)

Otherwise it is called an out-of-phase orbit. Here we study only in-phase orbits. They can be easily found from the uncoupled Hénon map because the coupling function \( g \) satisfies the condition (3).

Stability analysis of an in-phase orbit can be conveniently carried out [11,12] in a set of new coordinates \( (X_1, Y_1, X_2, Y_2) \) defined by
\[
\begin{align*}
X_1 &= \frac{(x_1 + x_2)}{2}, \quad Y_1 = \frac{(y_1 + y_2)}{2}, \\
X_2 &= \frac{(x_1 - x_2)}{2}, \quad Y_2 = \frac{(y_1 - y_2)}{2}.
\end{align*}
\] (5a, 5b)

Note that the in-phase orbit of the map (1) becomes the orbit of the new map (expressed in terms of new coordinates) with \( X_2 = Y_2 = 0 \). Moreover the new coordinates \( X_1 \) and \( Y_1 \) of the in-phase orbit also satisfy the uncoupled Hénon map.

Linearizing the new map at an in-phase orbit point, we obtain the Jacobian matrix \( J \) which decomposes into two \( 2 \times 2 \) matrices [11,12]:
\[
J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.
\] (6)

Here 0 is the \( 2 \times 2 \) null matrix, and
\[
J_1 = \begin{pmatrix} f'(X_1) & -1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} f'(X_1) - 2G(X_1) & -1 \\ 1 & 0 \end{pmatrix},
\] (7, 8)

where \( f'(X) = \frac{df}{dX} \) and \( G(X) \equiv \frac{\partial g(X_1,X_2)}{\partial X_2} \mid_{X_1=X_2=X} \).

Hereafter the function \( G(X) \) will be called the "reduced" coupling function of \( g(X_1,X_2) \). Note also that the determinant of each \( 2 \times 2 \) matrix \( J_i \) (\( i = 1, 2 \)) is one, i.e., \( \text{Det}(J_i) = 1 \). Hence they are area-preserving maps.

Stability of an in-phase orbit with period \( q \) is then determined from the \( q \) product \( M_i \) of the \( 2 \times 2 \) matrix \( J_i \):
\[
M_i \equiv \prod_{t=0}^{q-1} J_i(X_1(t)), \quad i = 1, 2.
\] (9)

Since \( \text{Det}(M_i) = 1 \), each matrix \( M_i \) has a reciprocal pair of eigenvalues, \( \lambda_i \) and \( \lambda_i^{-1} \). Associate with a pair of eigenvalues \( (\lambda_i, \lambda_i^{-1}) \) a stability index [16],
\[
\rho_i = \lambda_i + \lambda_i^{-1}, \quad i = 1, 2
\] (10)

which is just the trace of \( M_i \), i.e., \( \rho_i = \text{Tr}(M_i) \). Since \( M_i \) is a real matrix, \( \rho_i \) is always real. Note that the first stability index \( \rho_1 \) is just that for the case of the uncoupled Hénon map and hence coupling affects only the second stability index \( \rho_2 \).

An in-phase orbit is stable only when the moduli of its stability indices are less than or equal to two, i.e., \( |\rho_i| \leq 2 \) for \( i = 1 \) and 2. A period-doubling (tangent) bifurcation occurs when each stability index \( \rho_i \) decreases (increases) through \(-2 \) (2). Hence the stable region of the in-phase orbit in the parameter plane is bounded by four bifurcation lines associated with tangent and period-doubling bifurcations (i.e., those curves determined by the equations \( \rho_i = \pm 2 \) for \( i = 0, 1 \)). When the stability index \( \rho_1 \) decreases through \(-2 \), the in-phase orbit loses its stability via in-phase period-doubling bifurcation and gives rise to the birth of the period-doubled in-phase orbit. Here we are interested in scaling behaviors of such in-phase period-doubling bifurcations.

As an example we consider a linearly coupled case in which the coupling function is
\[
g(x_1, x_2) = \frac{c}{2}(x_2 - x_1).
\] (11)

Here \( c \) is a coupling parameter. As previously observed in Refs. [11,12], each "mother" stability region bifurcates into two "daughter" stability regions successively in the parameter plane. Thus the stable regions of in-phase orbits of period \( 2^n \) (\( n = 0, 1, 2, \ldots \)) form a "bifurcation" tree in the parameter plane [17].

An infinite sequence of connected stability branches (with increasing period) in the bifurcation tree is called a bifurcation "route" [11,12]. Each bifurcation route can be represented by its address, which is an infinite sequence of two symbols (e.g., \( L \) and \( R \)). A "self-similar" bifurcation "path" in a bifurcation route is formed by following a sequence of parameters \((\alpha_n, \gamma_n)\), at which the in-phase orbit of level \( n \) (period \( 2^n \)) has some given stability indices \((\rho_1, \rho_2)\) (e.g., \( \rho_1 = -2 \) and \( \rho_2 = 2 \)) [11,12].

All bifurcation paths within a bifurcation route converge to an accumulation point \((a^*, c^*)\), where the value of \( a^* \) is always the same as that of the accumulation point for the area-preserving case \((i.e., a^* = 4.1361668039404\ldots)\), but the value of \( c^* \) varies depending on the bifurcation routes. Thus each bifurcation route ends at a critical point \((a^*, c^*)\) in the parameter plane.

It has been numerically found that scaling behaviors near a critical point are characterized by two scaling factors, \( \delta_1 \) and \( \delta_2 \) [11,12]. The value of \( \delta_1 \) associated with scaling of the nonlinearity parameter is always the same as that of the scaling factor \( \delta (= 8.721\ldots) \) for the area-preserving case. However, the values of \( \delta_2 \) associated with scaling of the coupling parameter vary depending on the type of bifurcation routes. These numerical results agree well with analytic renormalization results [13], except for the case of one specific bifurcation route, called the \( E \) route. The address of the \( E \) route is \((L, R, \ldots)^\infty\) \((\equiv [L, R, L, R, \ldots])\) and it ends at the zero-coupling critical point \((a^*, 0)\).

Using an approximate renormalization method including truncation, Mao and Greene [13] obtained three relevant eigenvalues, \( \delta_1 = 8.9474, \delta_2 = -4.4510, \) and \( \delta_3 = 1.8762 \) for the zero-coupling case; hereafter the two eigenvalues \( \delta_2 \) and \( \delta_3 \) associated with coupling will be called the coupling eigenvalues (CE's). The two eigenvalues \( \delta_1 \) and \( \delta_2 \) are close to the numerical results of the nonlinearity-parameter scaling factor \( \delta_1 (= 8.721\ldots) \) and the coupling-parameter scaling factor \( \delta_2 (= -4.403\ldots) \) for the \( E \) route. However, they believed that the second relevant CE \( \delta_3 \) is an artifact of the truncation, because
it could not be identified with anything obtained by a
direct numerical method.

In order to resolve the discrepancy between the nu-
merical results and the renormalization results for the
zero-coupling case, we numerically reexamine the scaling
behavior associated with coupling. Extending the simple
one-term scaling law to a two-term scaling law, we find
a new scaling factor \( \delta_2 = 1.8505 \ldots \) associated with
coupling in addition to the previously found coupling scaling
factor \( \delta_2 = -4.4038 \ldots \), as will be seen below. The
values of these two coupling scaling factors are close to the
renormalization results of the relevant CE's \( \delta_2 \) and \( \delta_3 \).

We follow the in-phase orbits of period \( 2^n \) up to level
\( n = 14 \) in the \( E \) route and obtain a self-similar se-
quence of parameters \( (a_n, c_n) \), at which the pair of stability
indices, \( \rho_{0,n} \), \( \rho_{1,n} \), of the orbit of level \( n \) is \((-2,2)\).
The scalar sequences \( \{a_n\} \) and \( \{c_n\} \) converge geomet-
rically to their limit values, \( a^* \) and \( 0 \), respectively.
In order to see their convergence, define \( \delta_n = \Delta a_{n+1}/\Delta a_n \)
and \( \mu_n = \Delta c_{n+1}/\Delta c_n \), where \( \Delta a_n = a_n - a_{n-1} \) and
\( \Delta c_n = c_n - c_{n-1} \). Then they converge to their limit
values \( \delta \) and \( \mu \) as \( n \to \infty \), respectively. Hence the two
sequences \( \{a_n\} \) and \( \{c_n\} \) obey one-term scaling laws
asymptotically:

\[
\Delta a_n = C(a) \delta^{-n}, \quad \Delta c_n = C(c) \mu^{-n} \quad \text{for large } n, \tag{12}
\]

where \( C(a) \) and \( C(c) \) are some constants, \( \delta = 8.721 \ldots \),
and \( \mu = -4.403 \ldots \). The values of \( \delta \) and \( \mu \) are close to the
renormalization results of the first and second relevant
eigenvalues \( \delta_1 \) and \( \delta_2 \), respectively.

In order to take into account the effect of the second
relevant CE \( \delta_2 \) on the scaling of the sequence \( \{c_n\} \), we extend the simple one-term scaling law (12) to a two-term
scaling law:

\[
\Delta c_n = C_1 \mu_1^{-n} + C_2 \mu_2^{-n} \quad \text{for large } n, \tag{13}
\]

where \( \mu_1 > |\mu_2| \). This is a kind of multiple scaling law
[18]. Equation (13) gives

\[
\Delta c_n = t_1 \Delta c_{n+1} - t_2 \Delta c_{n+2}, \tag{14}
\]

where \( t_1 = \mu_1 + \mu_2 \) and \( t_2 = \mu_1 \mu_2 \). Then \( \mu_1 \) and \( \mu_2 \) are
solutions of the following quadratic equation:

\[
\mu^2 - t_1 \mu + t_2 = 0. \tag{15}
\]

To evaluate \( \mu_1 \) and \( \mu_2 \), we first obtain \( t_1 \) and \( t_2 \) from
\( \Delta c_n \)'s using Eq. (14):

\[
t_1 = \frac{\Delta c_n \Delta c_{n+1} - \Delta c_{n-1} \Delta c_{n+2}}{\Delta c_{n+1}^2 - \Delta c_n \Delta c_{n+2}}, \tag{16a}
\]

\[
t_2 = \frac{\Delta c_{n+1}^2 - \Delta c_{n+1} \Delta c_{n+2}}{\Delta c_{n+1}^2 - \Delta c_n \Delta c_{n+2}} \tag{16b}
\]

Note that Eqs. (13)-(16b) hold only for large \( n \). In fact
the values of \( t_k \)'s and \( \mu_i \)'s \( (i = 1,2) \) depend on the level
\( n \). Therefore we explicitly denote \( t_i \)'s and \( \mu_i \)'s by \( t_{i,n} \) and
\( \mu_{i,n} \), respectively. Then each of them converges to
a constant as \( n \to \infty \):

\[
\lim_{n \to \infty} t_{i,n} = t_i, \quad \lim_{n \to \infty} \mu_{i,n} = \mu_i, \quad i = 1,2. \tag{17}
\]

\begin{table}
\caption{Scaling factors \( \mu_{1,n} \) and \( \mu_{2,n} \) in the two-term scaling for the coupling parameter are shown in the second
and third columns, respectively. A product of them, \( \mu_{1,n}^2/\mu_{2,n} \), is shown in the fourth column.}
\begin{tabular}{cccc}
\hline
\n & \( \mu_{1,n} \) & \( \mu_{2,n} \) & \( \mu_{1,n}^2/\mu_{2,n} \) \\
\hline
5 & -4.403 989 128 & 10.473 4 & 1.850 17 \\
6 & -4.403 989 694 & 10.465 9 & 1.853 09 \\
7 & -4.403 988 736 & 10.458 2 & 1.854 46 \\
8 & -4.403 987 667 & 10.474 6 & 1.851 52 \\
9 & -4.403 987 847 & 10.473 9 & 1.851 68 \\
10 & -4.403 987 606 & 10.478 4 & 1.850 89 \\
11 & -4.403 987 607 & 10.478 6 & 1.850 85 \\
12 & -4.403 987 605 & 10.479 7 & 1.850 65 \\
\hline
\end{tabular}
\end{table}

Three sequences \( \{\mu_{1,n}\}, \{\mu_{2,n}\}, \text{ and } \{\mu_{1,n}/\mu_{2,n}\} \)
are shown in Table I. The second column shows rapid con-
vergence of \( \mu_{1,n} \) to its limit values \( \mu_1 = -4.403 987 805 \),
which is close to the renormalization result of the first rele-
cvant CE (i.e., \( \delta_2 = -4.4510 \)). From the third and fourth
columns, we also find that the second scaling factor \( \mu_2 \)
is given by a product of two relevant CE's \( \delta_2 \) and \( \delta_3 \),

\[
\mu_2 = \frac{\delta_2^2}{\delta_3}, \tag{18}
\]

where \( \delta_2 = \mu_1 \) and \( \delta_3 = 1.850 \, 65 \). It has been known that
every scaling factor in the multiple-scaling expansion of a
parameter is expressed by a product of the eigenvalues of
a linearized renormalization operator [18]. Note that the
value of \( \delta_3 \) is close to the renormalization result of the
second relevant CE (i.e., \( \delta_3 = 1.8762 \)).

We now study the coupling effect on the second stabil-
ity index \( \rho_{2,n} \) of the in-phase orbit of period \( 2^n \) near the
zero-coupling critical point \( (a^*,0) \). Figure 1 shows three
plots of \( \rho_{2,n}(a^*,c) \) versus \( c \) for \( n = 4,5, \) and \( 6 \). For \( c = 0 \),
\( \rho_{2,n} \) converges to a constant \( \rho_{2}^* (= -2.543 510 \ldots) \),
called the critical stability index [12], as \( n \to \infty \). How-
ever, when \( c \) is nonzero \( \rho_{2,n} \) diverges as \( n \to \infty \), i.e.,
its slope $S_n \equiv \frac{\partial^2 \rho_{2,n}}{\partial c \partial \sigma_{a,0}}$ at the zero-coupling critical point diverges as $n \to \infty$.

The sequence $\{S_n\}$ obeys a two-term scaling law,

$$S_n = D_1 \nu_1^n + D_2 \nu_2^n \quad \text{for large } n,$$

where $|\nu_1| > |\nu_2|$. This equation gives

$$S_{n+2} = r_1 S_{n+1} - r_2 S_n,$$  \(\text{(20)}\)

where $r_1 = \nu_1 + \nu_2$ and $r_2 = \nu_1 \nu_2$. As in the scaling for the coupling parameter, we first obtain $r_1$ and $r_2$ of level $n$ from $S_n$'s:

$$r_{1,n} = \frac{S_{n+1} S_n - S_{n+2} S_{n-1}}{S_n^2 - S_{n+1} S_{n-1}}, \quad r_{2,n} = \frac{S_{n+1}^2 - S_n S_{n+2}}{S_n^2 - S_{n+1} S_{n-1}}.$$  \(\text{(21)}\)

Then the scaling factors $\nu_{1,n}$ and $\nu_{2,n}$ of level $n$ are given by the roots of the quadratic equation, $\nu_1^2 - r_{1,n} \nu_1 + r_{2,n} = 0$. They are listed in Table II and converge to constants $\nu_1 (= -4.40389780550) \quad \text{and} \quad \nu_2 (= 1.8505335)$ as $n \to \infty$, whose accuracies are higher than those of the coupling-parameter scaling factors. Note that the values of $\nu_1$ and $\nu_2$ are also close to the renormalization results of the two relevant CE's $\delta_2$ and $\delta_3$.

We have also studied several other coupling cases with the coupling function, $g(x_1, x_2) = \frac{x_1^n}{2} (x_2^n - x_1^n)$ ($n$ is a positive integer). In all cases studied ($n = 2, 3, 4, 5$), the scaling factors of both the coupling parameter $c$ and the slope of the second stability index $\rho_2$ are found to be the same as those for the above linearly coupled case ($n = 1$) within numerical accuracy. Hence universality also seems to be well obeyed.

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### Table II. Scaling factors $\nu_{1,n}$ and $\nu_{2,n}$ in the two-term scaling for the slope of the second stability index are shown.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\nu_{1,n}$</th>
<th>$\nu_{2,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$-4.40389845359$</td>
<td>$1.8514335$</td>
</tr>
<tr>
<td>6</td>
<td>$-4.40389773029$</td>
<td>$1.8507826$</td>
</tr>
<tr>
<td>7</td>
<td>$-4.40389781385$</td>
<td>$1.8506036$</td>
</tr>
<tr>
<td>8</td>
<td>$-4.40389780407$</td>
<td>$1.8505538$</td>
</tr>
<tr>
<td>9</td>
<td>$-4.40389780521$</td>
<td>$1.8505400$</td>
</tr>
<tr>
<td>10</td>
<td>$-4.40389780507$</td>
<td>$1.8505361$</td>
</tr>
<tr>
<td>11</td>
<td>$-4.40389780509$</td>
<td>$1.8505350$</td>
</tr>
<tr>
<td>12</td>
<td>$-4.40389780509$</td>
<td>$1.8505349$</td>
</tr>
</tbody>
</table>

[14] See Table I in Ref. [13]. The $\delta_3$ in the text corresponds to $\delta'_3$ in the table.
[17] A "bifurcation" tree in the parameter plane is shown in Fig. 1 in each of Refs. [11,12].