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Universality of period doubling in coupled maps

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We study the critical behavior of period doubling in two coupled one-dimensional maps with a single maximum of order z. In particular, the effect of the maximum-order z on the critical behavior associated with coupling is investigated by a renormalization method. There exist three fixed maps of the period-doubling renormalization operator. For a fixed map associated with the critical behavior at the zero-coupling critical point, relevant eigenvalues associated with coupling perturbations vary, depending on the order z, whereas they are independent of z for the other two fixed maps. The renormalization results for the zero-coupling case are also confirmed by a direct numerical method.

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Universal scaling behavior of period doubling has been found in one-dimensional (1D) maps with a single maximum of order z ($z > 1$),

$$x_{i+1} = f(x_i) = 1 - A\,|x_i|^z, \quad z > 1. \tag{1}$$

For all $z > 1$, the 1D map (1) exhibits successive period-doubling bifurcations as the nonlinearity parameter A is increased. The period-doubling bifurcation points $A = A_n(z)$ ($n = 0, 1, 2, \ldots$), at which the nth period-doubling bifurcation occurs, converge to the accumulation point $A^*(z)$ on the A axis. The scaling behavior near the critical point A^* depends on the maximum-order z, i.e., the parameter and orbital scaling factors, δ and α, vary depending on z [1–4]. Therefore the order z determines universality classes.

Here we study the critical behavior of period doubling in a map T consisting of two identical 1D maps coupled symmetrically:

$$T: \begin{cases} x_{i+1} = F(x_i, y_i) = f(x_i) + g(x_i, y_i), \\ y_{i+1} = F(y_i, x_i) = f(y_i) + g(y_i, x_i), \end{cases} \tag{2}$$

where $f(x)$ is a 1D map (1) with a single maximum of even-order z ($z = 2, 4, 6, \ldots$) at $x = 0$, and $g(x, y)$ is a coupling function. The uncoupled 1D map f satisfies a normalization condition $f(0) = 1$, and the coupling function g obeys a condition $g(x, x) = 0$ for any x.

The quadratic-maximum case ($z = 2$) was previously studied in Refs. [5–9]. In this paper, using the renormalization method developed in Ref. [9], we extend the results for the $z = 2$ case to all even-order cases and investigate the dependence of the critical behavior associated with coupling on the order z.

The period-doubling renormalization transformation $\mathcal{N}$ for a coupled map T consists of squaring (T^2) and rescaling (B) operators:

$$\mathcal{N}(T) \equiv B T^2 B^{-1}. \tag{3}$$

Here the rescaling operator B is:

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \tag{4}$$

because we consider only in-phase orbits ($x_i = y_i$ for all i).

Applying the renormalization operator $\mathcal{N}$ to the coupled map (2) n times, we obtain the n-times renormalized map T_n of the form,

$$T_n: \begin{cases} x_{i+1} = F_n(x_i, y_i) = f_n(x) + g_n(x, y), \\ y_{i+1} = F_n(y_i, x_i) = f_n(y) + g_n(y, x). \end{cases} \tag{5}$$

Here f_n and g_n are the uncoupled and coupling parts of the n-times renormalized function F_n, respectively. They satisfy the following recurrence equations [9]:

$$f_{n+1}(x) = \alpha f_n\left(f_n\left(\frac{x}{\alpha}\right)\right), \tag{6}$$

$$\begin{aligned} g_{n+1}(x, y) = {} & \alpha f_n\left(f_n\left(\frac{x}{\alpha}\right) + g_n\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right)\right) \\ & + \alpha g_n\left(f_n\left(\frac{x}{\alpha}\right) + g_n\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), f_n\left(\frac{y}{\alpha}\right)\right. \\ & \left. + g_n\left(\frac{y}{\alpha}, \frac{x}{\alpha}\right)\right) - \alpha f_n\left(f_n\left(\frac{x}{\alpha}\right)\right), \end{aligned} \tag{7}$$

where the rescaling factor α is chosen to preserve the normalization condition $f_{n+1}(0) = 1$, i.e., $\alpha = 1/f_n(1)$. Equations (6) and (7) define a renormalization operator $\mathcal{R}$ for transforming a pair of functions (f, g);

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = \mathcal{R} \begin{pmatrix} f_n \\ g_n \end{pmatrix}. \tag{8}$$

A critical map T_c with the nonlinearity and coupling parameters set to their critical values is attracted to a fixed map T^* under iterations of the renormalization transformation $\mathcal{N}$,

$$T^*: \begin{cases} x_{i+1} = F^*(x_i, y_i) = f^*(x_i) + g^*(x_i, y_i), \\ y_{i+1} = F^*(y_i, x_i). \end{cases} \tag{9}$$

Here (f^*, g^*) is a fixed point of the renormalization operator $\mathcal{R}$ with $\alpha = 1/f^*(1)$, which satisfies $(f^*, g^*) = \mathcal{R}(f^*, g^*)$. Note that the equation for f^* is just the fixed-point equation in the 1D map case. The 1D fixed function f^* varies depending on the order z [4]. Consequently only the equation for the coupling fixed function g^* is left to be solved.

However it is not easy to directly solve the equation for the coupling fixed function. We therefore introduce a tractable recurrence equation for a "reduced" coupling function of the coupling function $g(x,y)$ [9], defined by

$$G(x) \equiv \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=x}. \tag{10}$$

Differentiating the recurrence equation (7) for g with respect to y and setting $y=x$, we have

$$G_{n+1}(x) = \left[f_n'\left(f_n\left(\frac{x}{\alpha}\right)\right) - 2G_n\left(f_n\left(\frac{x}{\alpha}\right)\right)\right] G_n\left(\frac{x}{\alpha}\right) + G_n\left(f_n\left(\frac{x}{\alpha}\right)\right) f_n'\left(\frac{x}{\alpha}\right). \tag{11}$$

Then Eqs. (6) and (11) define a "reduced" renormalization operator $\tilde{\mathcal{R}}$ for transforming a pair of functions (f, G):

$$\begin{pmatrix} f_{n+1} \\ G_{n+1} \end{pmatrix} = \tilde{\mathcal{R}} \begin{pmatrix} f_n \\ G_n \end{pmatrix}. \tag{12}$$

We look for a fixed point (f^*, G^*) of $\tilde{\mathcal{R}}$, which satisfies $(f^*, G^*) = \tilde{\mathcal{R}}(f^*, G^*)$. Here G^* is just the reduced coupling fixed function of g^* [i.e., $G^*(x) = \partial g^*(x,y)/\partial y|_{y=x}$]. As in the quadratic-maximum case ($z=2$) [9], we find three solutions for G^*:

$$G^*(x) = 0, \tag{13}$$

$$G^*(x) = \tfrac{1}{2} f^{*\prime}(x), \tag{14}$$

$$G^*(x) = \tfrac{1}{2}[f^{*\prime}(x) - 1]. \tag{15}$$

Here the first solution, corresponding to the reduced coupling fixed function of the zero-coupling fixed function $g^*(x,y) = 0$, is associated with the critical behavior at the zero-coupling critical point, whereas the second and third solutions dependent on the order z are associated with the critical behavior at other critical points [9].

Consider an infinitesimal reduced coupling perturbation $(0, \Phi(x))$ to a fixed point (f^*, G^*) of $\tilde{\mathcal{R}}$. We then examine the evolution of a pair of functions, $(f^*(x), G^*(x) + \Phi(x))$ under the reduced renormalization transformation $\tilde{\mathcal{R}}$. In the linear approximation we obtain a reduced linearized operator $\tilde{\mathcal{L}}$ of transforming a reduced coupling perturbation Φ:

$$\begin{aligned}\Phi_{n+1}(x) &= [\tilde{\mathcal{L}}\Phi_n](x) \\ &= \left[f^{*\prime}\left(f^*\left(\frac{x}{\alpha}\right)\right) - 2G^*\left(f^*\left(\frac{x}{\alpha}\right)\right)\right]\Phi_n\left(\frac{x}{\alpha}\right) \\ &\quad + \left[f^{*\prime}\left(\frac{x}{\alpha}\right) - 2G^*\left(\frac{x}{\alpha}\right)\right]\Phi_n\left(f^*\left(\frac{x}{\alpha}\right)\right). \end{aligned} \tag{16}$$

Here the prime denotes a derivative. If a reduced coupling perturbation $\Phi^*(x)$ satisfies

$$\nu\Phi^*(x) = [\tilde{\mathcal{L}}\Phi^*](x), \tag{17}$$

then it is called a reduced coupling eigenperturbation with coupling eigenvalue (CE) ν.

We first show that CE's are independent of the order z for the second and third solutions (14) and (15) of $G^*(x)$. In case of the second solution $G^*(x) = \frac{1}{2} f^{*\prime}(x)$, the reduced linearized operator $\tilde{\mathcal{L}}$ becomes a null operator, independently of z, because the right-hand side of Eq. (17) becomes zero. Therefore there exist no relevant CE's. For the third case $G^*(x) = \frac{1}{2}[f^{*\prime}(x) - 1]$, Eq. (17) becomes

$$\nu\Phi^*(x) = \Phi^*\left(\frac{x}{\alpha}\right) + \Phi^*\left(f^*\left(\frac{x}{\alpha}\right)\right). \tag{18}$$

When $\Phi^*(x)$ is a nonzero constant function, i.e., $\Phi^*(x) = c$ (c : nonzero constant), there exists a relevant CE, $\nu = 2$, independently of z.

In the zero-coupling case $G^*(x) = 0$, the eigenvalue equation (17) becomes

$$\nu\Phi^*(x) = f^{*\prime}\left(f^*\left(\frac{x}{\alpha}\right)\right)\Phi^*\left(\frac{x}{\alpha}\right) + f^{*\prime}\left(\frac{x}{\alpha}\right)\Phi^*\left(f^*\left(\frac{x}{\alpha}\right)\right). \tag{19}$$

Relevant CE's of Eq. (19) vary depending on the order z, as will be seen below.

An eigenfunction $\Phi^*(x)$ can be separated into two components, $\Phi^*(x) = \Phi^{*(1)}(x) + \Phi^{*(2)}(x)$ with $\Phi^{*(1)}(x) \equiv a_0^* + a_1^* x + \cdots + a_{z-2}^* x^{z-2}$ and $\Phi^{*(2)}(x) \equiv a_{z-1}^* x^{z-1} + a_z^* x^z + \cdots$, and the 1D fixed function f^* is a polynomial in x^z, i.e., $f^*(x) = 1 + c_z^* x^z + c_{2z}^* x^{2z} + \cdots$. Substituting the functions Φ^*, f^*, and $f^{*\prime}$ into the eigenvalue equation (19), it has the structure

$$\nu a_k^* = \sum_l M_{kl}(\{c^*\}) a_l^*, \quad k,l = 0,1,2,\ldots . \tag{20}$$

We note that each a_l^* ($l = 0,1,2,\ldots$) in the first and second terms in the right-hand side of Eq. (19) is involved only in the determination of coefficients of monomials x^k with $k = l + mz$ and $k = (z-1) + mz$ ($m = 0,1,2,\ldots$), respectively. Therefore any a_l^* with $l \geq z-1$ (in the right-hand side) cannot be involved in the determination of coefficients of monomials x^k with $k < z-1$, which implies that the eigenvalue equation (20) is of the form

$$\nu \begin{pmatrix} \Phi^{*(1)} \\ \Phi^{*(2)} \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ M_3 & M_2 \end{pmatrix} \begin{pmatrix} \Phi^{*(1)} \\ \Phi^{*(2)} \end{pmatrix}, \tag{21}$$

where M_1 is a $(z-1) \times (z-1)$ matrix, $\Phi^{*(1)} \equiv (a_0^*, \ldots, a_{z-2}^*)$, and $\Phi^{*(2)} \equiv (a_{z-1}^*, a_z^*, \ldots)$. From the reducibility of the matrix M into a semiblock form, it follows that to determine the eigenvalues of M it is sufficient to solve the eigenvalue problems for the two submatrices M_1 and M_2 independently.

We first solve the eigenvalue equation of M_1 ($\nu\Phi^{*(1)} = M_1\Phi^{*(1)}$), i.e.,

$$\nu a_k^* = \sum_l M_{kl}(\{c^*\}) a_l^*, \quad k,l = 0,\ldots,z-2. \tag{22}$$

Note that this submatrix M_1 is diagonal. Hence its eigenvalues are just the diagonal elements:

$$\nu_k = M_{kk} = \frac{f^{*\prime}(1)}{\alpha^k} = \alpha^{z-1-k}, \quad k = 0, \ldots, z-2. \quad (23)$$

Notice that all ν_k's are relevant eigenvalues.

Although ν_k is also an eigenvalue of M, $(\Phi_k^{*(1)}, 0)$ cannot be an eigenvector of M, because there exists a third submatrix M_3 in M [see Eq. (21)]. Therefore an eigenfunction $\Phi_k^*(x)$ in Eq. (19) with eigenvalue ν_k is a polynomial with a leading monomial of degree k, i.e., $\Phi_k^*(x) = \Phi_k^{*(1)}(x) + \Phi_k^{*(2)}(x) = a_k^* x^k + a_{z-1}^* x^{z-1} + a_z^* x^z + \cdots$, where $a_k^* \neq 0$.

We next solve the eigenvalue equation of M_2 ($\nu\Phi^{*(2)} = M_2\Phi^{*(2)}$), i.e.,

$$\nu a_k^* = \sum_l M_{kl}(\{c^*\}) a_l^*, \quad k, l = z-1, z, \ldots . \quad (24)$$

Unlike the case of M_1, $(0, \Phi^{*(2)})$ can be an eigenvector of M with eigenvalue ν. Then its corresponding function $\Phi^{*(2)}(x)$ is an eigenperturbation with eigenvalue ν, which satisfies Eq. (19). One can easily see that $\Phi^{*(2)}(x) = f^{*\prime}(x)$ is an eigenfunction with CE $\nu = 2$, which is the zth relevant CE in addition to those in Eq. (23). It is also found that there exist an infinite number of additional (coordinate change) eigenfunctions $\Phi^{*(2)}(x) = f^{*\prime}(x)[f^{*n}(x) - x^n]$ with irrelevant CE's α^{-n} ($n = 1, 2, \ldots$), which are associated with coordinate changes [10]. We conjecture that together with the z (noncoordinate change) relevant CE's, they give the whole spectrum of the reduced linearized operator $\tilde{\mathcal{L}}$ of Eq. (16) and the spectrum is complete.

Consider an infinitesimal coupling perturbation $g(x, y)$ [$= \varepsilon\varphi(x, y)$] to a critical map at the zero-coupling critical point, in which case the map T of Eq. (2) is of the form,

$$T: \begin{cases} x_{i+1} = F(x_i, y_i) = f_{A^*}(x_i) + \varepsilon\varphi(x, y), \\ y_{i+1} = F(y_i, x_i), \end{cases} \quad (25)$$

where the subscript A^* of f denotes the critical value of the nonlinearity parameter A and ε is an infinitesimal coupling parameter. The map T for $\varepsilon = 0$ is just the zero-coupling critical map consisting of two uncoupled 1D critical maps f_{A^*}. It is attracted to the zero-coupling fixed map (consisting of two 1D fixed maps f^*) under iterations of the renormalization transformation $\mathcal{N}$ of Eq. (3).

The reduced coupling function $G(x)$ of $g(x, y)$ is given by [see Eq. (10)]

$$G(x) = \varepsilon\Phi(x) \equiv \varepsilon \left.\frac{\partial\varphi(x, y)}{\partial y}\right|_{y=x}. \quad (26)$$

Then $\varepsilon\Phi(x)$ corresponds to an infinitesimal perturbation to the reduced zero-coupling fixed function $G^*(x) = 0$ of Eq. (13). The nth image Φ_n of Φ under the reduced linearized operator $\tilde{\mathcal{L}}$ of Eq. (16) has the form,

$$\Phi_n(x) = [\tilde{\mathcal{L}}^n \Phi](x) \simeq \sum_{k=0}^{z-2} \alpha_k \nu_k^n \Phi_k^*(x) + \alpha_{z-1} 2^n f^{*\prime}(x) \quad \text{for large } n, \quad (27)$$

since the irrelevant part of Φ_n becomes negligibly small for large n.

The stability multipliers $\lambda_{1,n}$ and $\lambda_{2,n}$ of the 2^n-periodic orbit of the map T of Eq. (25) are the same as those of the fixed point of the n-times renormalized map $\mathcal{N}^n(T)$ [9], which are given by

$$\lambda_{1,n} = f_n{}'(\hat{x}_n), \quad \lambda_{2,n} = f_n{}'(\hat{x}_n) - 2G_n(\hat{x}_n). \quad (28)$$

Here (f_n, G_n) is the nth image of (f_{A^*}, G) under the reduced renormalization transformation $\tilde{\mathcal{R}}$ [i.e., $(f_n, G_n) = \tilde{\mathcal{R}}^n(f_{A^*}, G)$], and $\hat{x}_n$ is just the fixed point of $f_n(x)$ [i.e., $\hat{x}_n = f_n(\hat{x}_n)$] and converges to the fixed point $\hat{x}$ of the 1D fixed map $f^*(x)$ as $n \to \infty$. In the critical case ($\varepsilon = 0$), $\lambda_{2,n}$ is equal to $\lambda_{1,n}$ and they converge to the 1D critical stability multiplier $\lambda^* = f^{*\prime}(\hat{x})$. Since $G_n(x) \simeq [\tilde{\mathcal{L}}_2^n G](x) = \varepsilon\Phi_n(x)$ for infinitesimally small ε, $\lambda_{2,n}$ has the form

$$\lambda_{2,n} \simeq \lambda_{1,n} - 2\varepsilon\Phi_n \simeq \lambda^* + \varepsilon\left[\sum_{k=0}^{z-2} e_k \nu_k^n + e_{z-1} 2^n\right] \quad \text{for large } n, \quad (29)$$

where $e_k = -2\alpha_k \Phi_k^*(\hat{x})$ ($k = 0, \ldots, z-2$) and $e_{z-1} = -2\alpha_{z-1} f^{*\prime}(\hat{x})$. Therefore the slope S_n of $\lambda_{2,n}$ at the zero-coupling point ($\varepsilon = 0$) is

$$S_n \equiv \left.\frac{\partial\lambda_{2,n}}{\partial\varepsilon}\right|_{\varepsilon=0} \simeq \sum_{k=0}^{z-2} e_k \nu_k^n + e_{z-1} 2^n \quad \text{for large } n. \quad (30)$$

Here the coefficients $\{e_k;\ k = 0, \ldots, z-1\}$ depend on the initial reduced function $\Phi(x)$, because the α_k's are determined only by $\Phi(x)$. Note that the magnitude of slope S_n increases with n unless all e_k's ($k = 0, \ldots, z-1$) are zero.

We choose monomials x^l ($l = 0, 1, 2, \ldots$) as initial reduced functions $\Phi(x)$, because any smooth function $\Phi(x)$ can be represented as a linear combination of monomials by a Taylor series. Expressing $\Phi(x) = x^l$ as a linear combination of eigenfunctions of $\tilde{\mathcal{L}}_2$, we have

$$\Phi(x) = x^l = \alpha_l \Phi_l^*(x) + \alpha_{z-1} f^{*\prime}(x) + \sum_{n=1}^{\infty} \beta_n f^{*\prime}(x)[f^{*n}(x) - x^n], \quad (31)$$

where α_l is nonzero for $l < z-1$ and zero for $l \geq z-1$, and all β_n's are irrelevant components. Therefore the slope S_n for large n becomes

$$S_n \simeq \begin{cases} e_l \nu_l^n + e_{z-1} 2^n & \text{for } l < z-1, \\ e_{z-1} 2^n & \text{for } l \geq z-1. \end{cases} \quad (32)$$

Note that the growth of S_n for large n is governed by two CE's ν_l and 2 for $l < z-1$ and by one CE $\nu = 2$ for $l \geq z-1$.

We numerically study the quartic-maximum case ($z = 4$) in the two coupled 1D maps (25) and confirm the renormalization results (32). In this case we follow the periodic orbits of period 2^n up to level $n = 15$ and obtain the slopes S_n of Eq. (30) at the zero-coupling critical point $(A^*, 0)$ ($A^* = 1.594\,901\,356\,228\ldots$) when the reduced function $\Phi(x)$ is a monomial x^l ($l = 0, 1, \ldots$).

The renormalization result implies that the slopes S_n for $l \geq z-1$ obey a one-term scaling law asymptotically:

$$S_n = d_1 r_1^n. \tag{33}$$

We therefore define the growth rate of the slopes as follows:

$$r_{1,n} \equiv \frac{S_{n+1}}{S_n}. \tag{34}$$

Then it will converge to a constant r_1 as $n \to \infty$. A sequence of $r_{1,n}$ for $\Phi(x) = x^3$ is shown in the second column of Table I. Note that it converges fast to $r_1 = 2$. We have also studied several other reduced-coupling cases with $\Phi(x) = x^l$ $(l > 3)$. In all higher-order cases studied, the sequences of $r_{1,n}$ also converge fast to $r_1 = 2$.

When $l < z-1$, two relevant CE's govern the growth of the slopes S_n. We therefore extend the simple one-term scaling law (33) to a two-term scaling law:

$$S_n = d_1 r_1^n + d_2 r_2^n, \quad \text{for large } n, \tag{35}$$

where $|r_1| > |r_2|$. This is a kind of multiple-scaling law [11]. Equation (35) gives

$$S_{n+2} = t_1 S_{n+1} - t_2 S_n, \tag{36}$$

where $t_1 = r_1 + r_2$ and $t_2 = r_1 r_2$. Then r_1 and r_2 are solutions of the following quadratic equation,

$$r^2 - t_1 r + t_2 = 0. \tag{37}$$

To evaluate r_1 and r_2, we first obtain t_1 and t_2 from the S_n's using Eq. (36):

$$t_1 = \frac{S_{n+1}S_n - S_{n+2}S_{n-1}}{S_n^2 - S_{n+1}S_{n-1}}, \quad t_2 = \frac{S_{n+1}^2 - S_n S_{n+2}}{S_n^2 - S_{n+1}S_{n-1}}. \tag{38}$$

Note that Eqs. (35)–(38) are valid for large n. In fact, the values of t_i's and r_i's $(i = 1, 2)$ depend on the level n. Thus we denote the values of t_i's in Eq. (38) explicitly by $t_{i,n-1}$'s, and the values of r_i's obtained from Eq. (37) are also denoted by $r_{i,n-1}$'s. Then each of them converges to a constant as $n \to \infty$:

$$\lim_{n\to\infty} t_{i,n} = t_i, \quad \lim_{n\to\infty} r_{i,n} = r_i, \quad i = 1, 2. \tag{39}$$

TABLE I. In the quartic-maximum case ($z = 4$), a sequence $\{r_{1,n}\}$ for a one-term scaling law is shown in the second column when $\Phi(x) = x^3$, and two sequences $\{r_{1,n}\}$ and $\{r_{2,n}\}$ for a two-term scaling law are shown in the third and fourth columns when $\Phi(x) = 1$.

	$\Phi(x) = x^3$	$\Phi(x) = 1$	
n	$r_{1,n}$	$r_{1,n}$	$r_{2,n}$
5	1.999 920 2	-4.829 455 8	1.958
6	2.000 009 3	-4.829 422 6	2.090
7	1.999 994 7	-4.829 409 8	1.973
8	2.000 000 4	-4.829 406 8	2.039
9	1.999 999 6	-4.829 405 8	1.984
10	2.000 000 0	-4.829 405 5	2.018
11	2.000 000 0	-4.829 405 5	1.992
12	2.000 000 0	-4.829 405 4	2.009

The two-term scaling law (35) is very well obeyed. Sequences $r_{1,n}$ and $r_{2,n}$ for $\Phi(x) = 1$ are shown in the third and fourth columns of Table I. They converge fast to $r_1 = \alpha^3$ $(\alpha = -1.6903\ldots)$ and $r_{2,n} = 2$, respectively. We have also studied two other reduced-coupling cases with $\Phi(x) = x^l$ $(l = 1, 2)$. It is found that the sequences $r_{1,n}$ and $r_{2,n}$ for $l = 1(2)$ converge fast to their limit values $r_1 = \alpha^2(2)$ and $r_2 = 2(\alpha)$, respectively.

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