Singularity spectrum for period n-tupling in area-preserving maps

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The singularity spectrum $f(\alpha)$ and the generalized dimension $D(q)$ of the critical orbit is numerically obtained to study the global scaling behavior of period n-tupling ($n=2,3,4$) in area-preserving maps. It is found that $f(\alpha)$ becomes quite different as n is changed, and the generalized dimension $D(q)$ increases for all q as n increases. The global scaling behavior of conservative systems is different from that of dissipative systems. Moreover, for conservative systems, the global scaling behavior seems to depend on dimensionality.

I. INTRODUCTION

Recently, Halsey *et al.*[1] introduced the singularity spectrum $f(\alpha)$ to describe the global scaling behavior of the probability measure lying on strange sets. This formalism has been applied to many systems, for example, period doubling in the logistic map,[1] the devil's staircase of mode locking and the critical "golden-mean" orbit of the circle map,[1] the spectrum of a quasiperiodic Schrödinger operator,[2] and the critical KAM curve in the standard map.[3]

The purpose of this paper is to report on the application of the formalism to period n-tupling in the area-preserving De Vogelaere map. The De Vogelaere map T_p is of the following form:

$$T_p:\quad \begin{aligned} x_{n+1}&=-y_n+f_p(x_n)\,,\\ y_{n+1}&=x_n-f_p(x_{n+1})\,, \end{aligned} \tag{1.1}$$

where $f_p(x)=px-(1-p)x^2$. The local scaling behavior about some special point has been studied for period doubling[4-8] and higher period n-tupling.[8-11] They[8,9] found that the pattern of periodic orbits repeats itself asymptotically from one bifurcation to the next for even n, and to every other one for odd n. In other words, for even period n-tupling, the pattern of periodic orbits exhibits a "period-1" scaling behavior; whereas for odd period n-tupling, it exhibits a "period-2" scaling behavior. The scaling behavior of the power spectrum[12] and the trajectory scaling function of the displacement vectors[13] were studied for period doubling. However, for higher period n-tupling ($n\geq 3$), the global scaling behavior is still unknown. It is therefore interesting to study the singularity spectrum of the critical orbit for period n-tupling and see how the singularity spectrum and the generalized dimension change as n is varied. In Sec. II we obtain the singularity spectra for period n-tupling ($n=2,3,4$) by the ratio method, and compare them. In Sec. III we summarize the results.

II. SINGULARITY SPECTRUM FOR PERIOD n-TUPLING

We now use the algorithm developed in Ref. 1 to calculate the singularity spectrum. We take the critical orbit of level m whose period N is n^m. Let

$$z_k^{(m)}=(x_k^{(m)},y_k^{(m)})=T_{p^*}^k(z_0^{(m)}),\quad k=0,1,\dots,n^{m-1}\,.$$

Here p^* is the accumulation point for period n-tupling[6,8,9]

$$\begin{aligned} p^*&=-1.266\,311\,276\,922\,10\quad (n=2)\\ &=-0.477\,013\,684\,274\,04\quad (n=3)\\ &=-0.689\,824\,440\,283\,47\quad (n=4)\,. \end{aligned}$$

At the accumulation point p^* and for each level m, there is one unstable orbit for $n=2$ whose critical residue value R^* is 1.135 87. However, for higher n ($n\geq 3$), there exist two kinds of orbits, one stable and the other unstable.[9,11] The critical residue values of the stable and unstable orbits for $n=3$ and 4 are, respectively,

$$\begin{aligned} R^*&=0.7337,\ -0.0092\ (n=3)\\ &=0.5178,\ -0.0277\ (n=4)\,. \end{aligned}$$

We then define the partition of level m as follows. For $n=2$, let the length l_k be the distance between $z_k^{(m)}$ and its closest point $z_{k+N/2}^{(m)}$:

$$l_k=z_{k+N/2}^{(m)}-z_k^{(m)}\,. \tag{2.1}$$

These lengths serve as natural scales for a partition of level m with probability measure $M_k=2/N$ attributed to each scale. For $n=3$, however, the closest points to $z_k^{(m)}$ are $z_{k+N/3}^{(m)}$ and $z_{k+2N/3}^{(m)}$. Let us denote the triangle whose vertices are $z_k^{(m)}$, $z_{k+N/3}^{(m)}$, and $z_{k+2N/3}^{(m)}$ by $T_k^{(m)}$. Then there are $N/3$ triangles $T_k^{(m)}$ ($k=0,1,\dots,N/3-1$). For each triangle $T_k^{(m)}$, there are three sides, and let the length $l_{k,j}$ be the length of the jth side:

$$l_{k,j}=z_{k+jN/3}^{(m)}-z_{k+(j-1)N/3}^{(m)}\quad (j=1,2,3)\,. \tag{2.2}$$

Then for $n=3$, these lengths serve as natural scales for a partition of level m with probability measure $M_{k,j}=1/N$ attributed to each scale. Similarly, for $n=4$, there are three points $z_{k+N/4}^{(m)}$, $z_{k+2N/4}^{(m)}$, and $z_{k+3N/4}^{(m)}$ which are closest to $z_k^{(m)}$. Let us denote the quadrilateral whose vertices are $z_k^{(m)}$, $z_{k+N/4}^{(m)}$, $z_{k+2N/4}^{(m)}$, and $z_{k+3N/4}^{(m)}$ by $Q_k^{(m)}$. Then there are $N/4$ quadrilaterals $Q_k^{(m)}$

($k=0,1,\ldots,N/4-1$). Let the length $l_{k,j}$ be the length of the jth side of the kth quadrilateral:

$$l_{k,j}=z^{(m)}_{k+jN/4}-z^{(m)}_{k+(j-1)N/4} \quad (j=1,2,3,4)\ . \tag{2.3}$$

Then, like the period-tripling case, these lengths serve as natural scales for a partition of level m with probability measure $M_{k,j}=1/N$ attributed to each scale.

The probability measure $M_i(l_i)$ can be described by defining a scaling index α_i of the form[1]

$$M_i(l_i)=l^{\alpha_i(l_i)}\ . \tag{2.4}$$

Typically, for sufficiently small l_i, the scaling index α_i takes on a range of values between $\alpha_{\rm min}$ and $\alpha_{\rm max}$. Let $n(\alpha)d\alpha$ be the number of singularities of type α' for all α' lying between α and $\alpha+d\alpha$, and l a typical length of the partition. Then the number of singularities can be described by defining a scaling index $f(\alpha)$ of the form[1]

$$n(\alpha)d\alpha=\rho(\alpha)l^{-f(\alpha)}d\alpha\ . \tag{2.5}$$

$f(\alpha)$ may be interpreted as the "dimension" of the subset of singularities of type α.

Now we want to know the possible values of α and the function $f(\alpha)$ for period n-tupling. With the partition of level m defined above, we form the partition function $Z_m(q,\tau)$

$$\begin{aligned} Z_m(q,\tau)&=\sum_{k=0}^{N/2-1} M_k^q l_k^{-\tau} \quad (n=2)\\ &=\sum_{k=0}^{N/3-1}\sum_{j=1}^{3} M_{k,j} l_{k,j}^{-\tau} \quad (n=3)\\ &=\sum_{k=0}^{N/4-1}\sum_{j=1}^{4} M_{k,j}^q l_{k,j}^{-\tau} \quad (n=4)\ . \end{aligned} \tag{2.6}$$

Halsey *et al.*[1] argued that for large m there is a unique critical function $\tau_c(q)$ such that

$$z_m(q,\tau)=\begin{cases}\infty & \text{for } \tau>\tau_c(q)\\ 0 & \text{for } \tau<\tau_c(q)\ .\end{cases} \tag{2.7}$$

That is, the partition function $Z_m(q,\tau)$ is of order unity only when $\tau=\tau_c(q)$. The critical function $\tau_c(q)$ is related to the generalized dimension D_q of Hentschel and Procaccia[12] by

$$(q-1)D(q)=\tau_c(q)\ . \tag{2.8}$$

For example, $D(0)$ is the Hausdorff dimension of the support of the probability measure, while $D(1)$ is the information dimension and $D(2)$ the correlation dimension.[14] The scaling indices α and $f(\alpha)$ are given by a Legendre transformation

$$\alpha(q)=\frac{d\tau_c(q)}{dq},\quad f(q)=q\alpha(q)-\tau_c(q)\ . \tag{2.9}$$

Eliminating q gives us the singularity spectrum $f(\alpha)$ defined in a range $[\alpha_{\rm min},\alpha_{\rm max}]$.

In practice, the solutions to $Z_m(q,\tau_c)=1$ converge slowly with m. To improve the convergence, we use the ratio method.[1] We can determine $\tau_c(q)$ by requiring that

$$Z_{m+j}(q,\tau_c)/Z_m(q,\tau_c)=1$$

for some j which depends on period n-tupling. As mentioned in Sec. I, even period n-tupling sequences exhibit a "period-1" scaling behavior, while odd-period n-tupling sequences exhibit a "period-2" scaling behavior.[8,9] Therefore, for even-period n-tupling, $j=1$; whereas for odd-period n-tupling, $j=2$. After obtaining τ_c, we compute α algebraically rather then differentiate τ_c with respect to q. After some algebra, we obtain

$$\alpha=\frac{\langle \ln(M)\rangle_m-\langle \ln(M)\rangle_{m+j}}{\langle \ln(l)\rangle_m-\langle \ln(l)\rangle_{m+j}}\ , \tag{2.10}$$

where $\langle F\rangle_m$ denotes the average value of F with respect to the probability distribution $P_i(m)$ of level m,

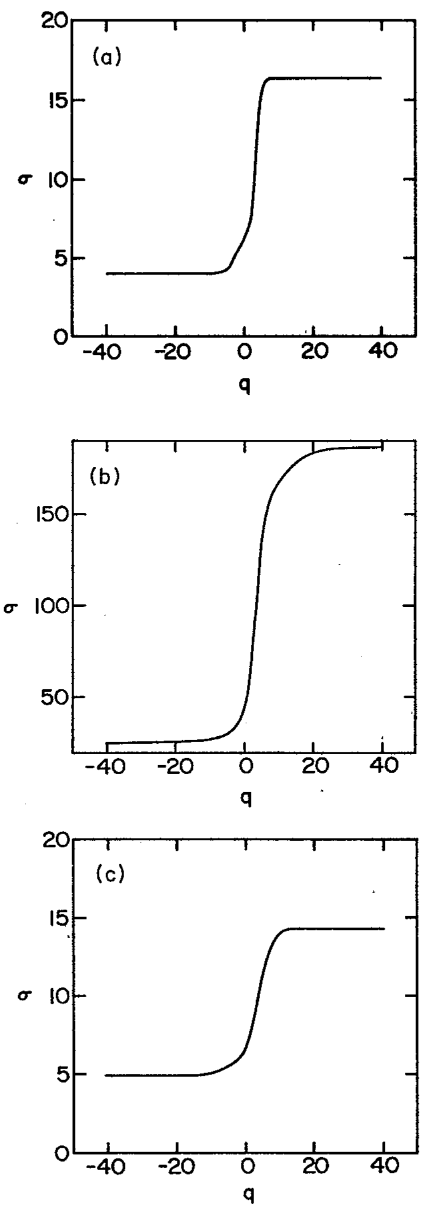

FIG. 1. The scaling function $\sigma(q)$ for period n-tupling. (a) $n=2$, (b) $n=3$, (c) $n=4$.

$$P_i(m)=\frac{M_i^q l_i^{-\tau_c}}{Z_m(q,\tau_c)} . \quad (2.11)$$

Let us define the typical length $L(m)$ of level m by

$$\ln[L(m)]=\langle \ln(l)\rangle_m . \quad (2.12)$$

Then, since the probability measure M_i is constant for period n-tupling, the scaling index α is given by

$$\alpha=\frac{\ln(n^j)}{\ln[L(m)/L(m+j)]} . \quad (2.13)$$

Let us define the scaling function $\sigma(q)$ as the ratio of the typical length of level m to that of level $m+j$,

$$\sigma(q)=\frac{L(m)}{L(m+j)} . \quad (2.14)$$

Then the scaling index α becomes

$$\alpha=\frac{\ln(n^j)}{\ln[\sigma(q)]} . \quad (2.15)$$

From Eq. (2.15), we see that by varying q we can visit the different regions with singularities of type α. The maximum and minimum values of the scaling function σ_{max} and σ_{min} determine the range $[\alpha_{min},\alpha_{max}]$, and hence the range of $D(q)$

$$\alpha_{max}=D(-\infty)=\frac{\ln(n^j)}{\ln(\sigma_{min})} ,$$
$$\alpha_{min}=D(\infty)=\frac{\ln(n^j)}{\ln(\sigma_{max})} . \quad (2.16)$$

Here α_{min} corresponds to the region in the set where the probability measure is the most concentrated, while α_{max} corresponds to the region where the probability measure is the most rarefied.

The scaling functions $\sigma(q)$ for period n-tupling ($n=2,3,4$) are shown in Fig. 1. From σ_{max} and σ_{min}, we obtain the range

$$[\alpha_{min},\alpha_{max}]=[0.248,0.498] \quad (n=2)$$
$$=[0.420,0.680] \quad (n=3)$$
$$=[0.522,0.873] \quad (n=4) .$$

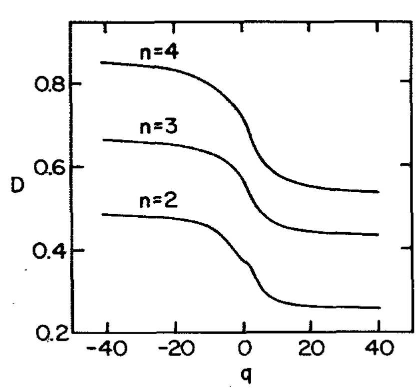

FIG. 2. The generalized dimension $D(q)$ for period n-tupling ($n=2,3,4$).

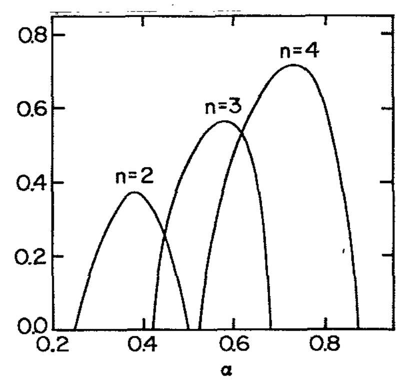

FIG. 3. The singularity spectrum $f(\alpha)$ for period n-tupling ($n=2,3,4$).

For $n=2$, the values of σ_{min} and σ_{max} are equal to the local scaling factors α and β,[6,8] where α is the scaling factor along the symmetry line and β the scaling factor across the symmetry line. However, for $n=3$ and 4, only σ_{max} is equal to the local scaling factor β across the symmetry line.[8,9] This means that in some other region which is not near a symmetric point, the probability measure is the most rarefied for $n=3$ and 4.

After obtaining τ_c and α, we can compute the generalized dimension $D(q)$ and the singularity spectrum $f(\alpha)$ from Eqs. (2.8) and (2.9). The numerical result of $D(q)$ and $f(\alpha)$ are shown in Figs. 2 and 3. We see that $D(q)$ is a decreasing function of q,[12] and $f(\alpha)$ is a convex function with a single maximum at $q=0$.[1] The maximum value of $f(\alpha)$ is just the Hausdorff dimension. For $n=2$, $f(\alpha)$ is completely different from that of period doubling in the logistic map.[1] Therefore, the global scaling behavior of conservative maps is different from that of dissipative maps. Furthermore, the left part of $f(\alpha)$ which corresponds to the region in which q varies from 0 to ∞ is quite different from that of four-dimensional symplectic maps,[15] whereas the right part is only slightly different. Therefore, the global scaling behavior of period doubling in symplectic maps seems to depend on dimensionality. For $n=3$ and 4, there are, as mentioned previously, two critical orbits, one stable and the other unstable. We have studied the two critical orbits and found that the singularity spectra of the two orbits agree very well. Finally, we have studied how the singularity spectrum and the generalized dimension for period n-tupling change as n is varied. As shown in Figs. 2 and 3, the singularity spectra for different period n-tupling are quite different, and the generalized dimension increases for all q as n is changed. Therefore, different period n-tupling shows different global scaling behavior.

In a recent study of the universal scaling ratio of the power spectrum in period n-tupling, Hu, Shi, and Kim[12] observed that the scaling ratio increases with n. However, the *rate* of increase seems to slow down and approach a limiting value. It will be interesting to see if similar behavior occurs for $f(\alpha)$ and $D(q)$.

III. SUMMARY

We have obtained the singularity spectrum and the generalized dimension of period n-tupling ($n=2,3,4$) in area-preserving maps to study the global scaling behavior of the critical orbit. It is found that the singularity spectrum $f(\alpha)$ for period n-tupling becomes quite different as n is changed, and the generalized dimension increases for all q as n increases. The singularity spectrum of period doubling in area preserving maps is different from those of the logistic map and four-dimensional symplectic maps. Therefore, the global scaling behavior of conservative systems is different from that of dissipative systems. Moreover, for conservative systems, the global scaling behavior seems to depend on dimensionality.

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