SCALING BEHAVIORS IN THE RESONANCE MULTIFURCATIONS IN THE 2-D AREA-PRESERVING MAPS

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ABSTRACT

The resonance multifurcation sequences in the 2-D area-preserving maps are asymptotically self similar. The similar pattern of periodic orbits repeats itself asymptotically from one multifurcation to the next for even n and to every other for odd n. The renormalization group method can be applied to these phenomena to obtain the multifurcation ratios and scaling factors. They agree well with the numerically calculated values following the multifurcation sequences.

1. INTRODUCTION

When a sufficiently small nonlinear perturbation is applied to an integrable system, phase space is generally divided into two regions, chaotic and regular. Generically the regular region comprises distorted invariant tori survived from the integrable system and islands surrounding elliptic periodic points. Chaotic region consists of homoclinic orbits around hyperbolic periodic orbits. These elliptic and hyperbolic orbits are born out of destroyed tori of integrable system satisfying the resonance condition due to the Poincare-Birkoff theorem. As the perturbation strength is increased the invariant tori are gradually destroyed at first while the island structures as well as chaotic region grow. Invariant tori are broken as if squeezed by the nearby growing island structures as well as homoclinic orbits. As the perturbation strength is further increased, island structures decay eventually through the route of period doubling cacade contributing to the growing size of the chaotic region. For this reason the last surviving invariant torus seems the furthermost one from the periodic points which accompany either island structure or chaotic region.

The island structure also comprises primary invariant curves and secondary island structures with their counterpart homoclinic orbits.

In general secondary structures are emanated from the central periodic orbits whenever the central elliptic periodic point satisfies certain resonance condition. The secondary structures are similar to the primary structures in that islands are invariant curves surrounding elliptic periodic points interwoven with chaotic region consisting of homoclinic orbits. These structures also accompany tertiary island structures similar to the secondary structures. This process goes on and the island structures are infinitely nested.

Generally invariant tori survived from the integrable system prevents the phase space becoming extended chaos. However if the perturbation strength passes certain critical strength the last surviving torus is destroyed and the extended chaos sets in. At this point on major hindrance to the transport in the Hamiltonian system becomes the island structures. The area of island structure decreases smaller as the perturbation strength increases and the probability of trapped orbit inside islands becomes negligibly small. One might expect that almost any orbit in the phase space can freely wander around any point in the chaotic region. However this is far from the true picture. Numerical experiments suggest that longtime correleations are strongly influenced by the existence of island structures[1]. Instead of free wandering, orbits in the chaotic region are frequently stuck to the outer edge of island structures and stay stuck for long time.

In this paper we discuss scaling behaviors of infinitely nested island structures. These nested structures are self similar and renormalization group (RG) analysis can be carried out to obtain the universal map (the fixed point map in the RG space). Actually the self similarity is truely good only in the vicinity of periodic point. This fact is indeed utilized in the implementation of RG technique.

2. RENORMALIZATION GROUP TRANSFORMATION

The RG method we use here may be called as the method of quadratic approximants similar to that of Helleman[2]. We first construct from the original map T, T^{n^ℓ}, $T^{n^{\ell+1}}$ and Taylor expand them keeping upto quadratic terms. The RG transformation equations are derived comparing the coefficients of similar terms. The reason we keep quadratic terms is that although linear terms yield quite lot of information on the scaling behavior including ratio α/β of scaling factors α and β, they are necessary to determine these factors seperately.

Since all quadratic maps are equivalent[3], we use the Devogelaere map for the RG analysis since the map is represented in terms of symmetry coordinate. This map is given as:

$$T_p : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' = -y + f_p(x) \\ y' = x - f_p(x') \end{pmatrix}, \quad f_p(x) = px - (1-p)x^2 . \quad (1)$$

The Devogelaere map is area-preserving and also reversible. We consider only those periodic points which lie on the symmetry line which is xaxis in this case.

We illustrate the method by taking n=2. Let us denote the symmetric periodic point of period 2^{ℓ} as $(\hat{x}_{\ell},0)$. The idea is to associate, for each value p', a value p such that $T_p^{2^{\ell}}$ with origin $(\hat{x}_{\ell},0)$ looks the same as $T_{p'}^{2^{\ell-1}}$ the origin shifted to $(\hat{x}_{\ell-1},0)$ on a small spatial scale. Therefore,

$$T_{p'}^{2^{(\ell-1)}} = \Lambda T_p^{2^{\ell}} \Lambda^{-1}, \qquad \Lambda = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}. \tag{2}$$

We now Taylor expand $T_p^{2^{\ell}}$'s about fixed points retaining only upto quadratic terms. We thus write as

$$\begin{aligned} F_p^{(\ell)}(x,y) &= \hat{x}_{\ell} + A_{\ell}(p)(x-\hat{x}_{\ell}) + B_{\ell}(p)y + U_{\ell}(p)\ (x-\hat{x}_{\ell})^2 \\ &\quad + V_{\ell}(p)\cdot(x-\hat{x}_{\ell})\cdot y + W_{\ell}(p)y^2 + \cdots\cdots, \\ G_p^{(\ell)}(x,y) &= C_{\ell}(p)(x-\hat{x}_{\ell}) + D_{\ell}(p)\cdot y + Q_{\ell}(p)\cdot(x-\hat{x}_{\ell})^2 \\ &\quad + R_{\ell}(p)\cdot(x-\hat{x}_{\ell})\cdot y + S_{\ell}(p)\cdot y^2 + \cdots\cdots . \end{aligned} \tag{3}$$

The eq.(2) now yeilds the recursion relations

$$\begin{aligned} &A_{\ell-1}(p') = A_{\ell}(p),\ D_{\ell-1}(p') = D_{\ell}(p),\ B_{\ell-1}(p') = \frac{\alpha}{\beta}B_{\ell}(p), \\ &C_{\ell-1}(p') = \frac{\beta}{\alpha}C_{\ell}(p),\ U_{\ell-1}(p') = \frac{1}{\alpha}U_{\ell}(p),\ V_{\ell-1}(p') = \frac{1}{\beta}V_{\ell}(p), \\ &W_{\ell-1}(p') = \frac{\alpha}{\beta^2}W_{\ell}(p),\ Q_{\ell-1}(p') = \frac{\beta}{\alpha^2}Q_{\ell}(p),\ R_{\ell-1}(p') = \frac{1}{\alpha}R_{\ell}(p) \end{aligned} \tag{4}$$

and $S_{\ell-1}(p') = \frac{1}{\beta}S_{\ell}(p)$.

We next define the linearized map $M_p^{(\ell)}(x,y)$ of $T_p^{2^{\ell}}(x,y)$ as

$$M_p^{(\ell)}(x,y) = DT_p^{2^{\ell}}(x,y) = \begin{pmatrix} H_p^{(\ell)}(x,y) & I_p^{(\ell)}(x,y) \\ J_p^{(\ell)}(x,y) & K_p^{(\ell)}(x,y)\end{pmatrix}. \tag{5}$$

These four new functions are not really independent but related among themselves through the area preserving condition. The derivatives of these functions at the periodic points (the origin of the coordinate) also yield the coefficients of quadratic terms in the expansion, eq.(3).

The accumulation parameter value p^* can be obtained by locating zero of the following function, i.e.

$$F(p) \equiv \text{Tr}\ M_{\ell}(p) = \text{Tr}\ M_{\ell-1}(p) = 0. \tag{6}$$

The bifurcation ratio is the ratio of the slopes of $\text{Tr}\ M_{\ell}(p)$ and $\text{Tr}\ M_{\ell-1}(p')$ at the accumulation parameter value p^*. Therefore we have dp |p* dp |p*/ dp' |p* . (7)

Once we have p*, we can obtain the scaling factors α and β through the eq. (4).

Since the universal map T^* is the fixed point map in the RG space, i.e., T^{n^ℓ} in the limit $\ell \to \infty$, we can obtain better estimates of T^*'s as we increase ℓ.

3. RESULTS OF THE RG ANALYSIS

We have performed RG analysis for n=2 ...... 6 with various ℓ. The results are given in the following table. They agree well with the values calculated following the multifurcation sequences.

m/n-resonance	ℓ	p*	δ	α	β
1/2	6	-1.2663112769223	8.721090	- 4.01806	16.36386
1/3	3	- .477013684274045	407.4254	-43.9794	-186.723
1/4	5	- .0689824440291	24.4616	- 5.6119	14.2824
1/5	2	.177137427506	401.75	-43.34	- 76.09
1/6	5	.3362383932	13.83	- 8.248	6.302

As was reported in our previous work[4)], the self similarity is recovered every multifurcation for even n and every other times for odd n. Therefore there are two sets of universal maps for odd n although products of scaling factors of successive multifurcations are same in either case. Another point worth to mention is that unlike in the case of n=2, the residue of the universal map is less than unity which implies that the fixed point at the accumulation parameter value is elliptic.

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