Bicritical Behavior in Unidirectionally Coupled Systems

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Abstract. We study the scaling behavior of period doublings in two unidirectionally coupled one-dimensional maps near a bicritical point where two critical lines of period-doubling transition to chaos in both subsystems meet. Note that the bicritical point corresponds to a border of chaos in both subsystems. For this bicritical case, the second response subsystem exhibits a new type of non-Feigenbaum critical behavior, while the first drive subsystem is in the Feigenbaum critical state. In order to make an analysis of the bicritical behavior, we develop a new version of the renormalization group method based on the eigenvalue matching, and obtain the bicritical point, the parameter and orbital scaling factors with remarkably high numerical precision. These scaling results obtained in the abstract system are also confirmed in the real system of two parametrically forced pendulums with a one-way coupling.

Period-doubling transition to chaos has been extensively studied in a one-parameter family of one-dimensional (1D) unimodal maps,

$$x_{t+1} = 1 - Ax_t^2, \tag{1}$$

where x_t is a state variable at a discrete time t. As the control parameter A is increased, the 1D map undergoes an infinite sequence of period-doubling bifurcations accumulating at a critical point A_c, beyond which chaos sets in. Using a renormalization group (RG) method, Feigenbaum [1] has discovered universal scaling behavior near the critical point A_c.

Here we are interested in the period doublings in a system consisting of two 1D maps with a one-way coupling,

$$x_{t+1} = 1 - Ax_t^2, \;\; y_{t+1} = 1 - By_t^2 - Cx_t^2, \tag{2}$$

where x and y are state variables of the first and second subsystems, A and B are control parameters of the subsystems, and C is a coupling parameter. Note that

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the first (drive) subsystem acts on the second (response) subsystem, while the second subsystem does not influence the first subsystem. This kind of unidirectionally coupled 1D maps have been used as a model for open flow systems [2]. In particular, such systems with unidirectional coupling are actively discussed recently in application to secure communication using synchronous chaos [3].

A new kind of non-Feigenbaum scaling behavior was first found in the unidirectionally-coupled 1D maps (2) near a bicritical point (A_c, B_c) where two critical lines of period-doubling transition to chaos in both subsystems meet [4]. For this bicritical case, a RG analysis was also developed and the corresponding fixed point, governing the bicritical behavior, was numerically obtained by directly solving the RG fixed-point equation using a polynomial approximation [5]. In this paper, we develop a new version of the RG approach based on the eigenvalue matching, and make an analysis of the bicriticality. Thus we numerically obtain the bicritical point, the parameter and orbital scaling factors with remarkably high numerical precision. In order to confirm the scaling results obtained in the unidirectionally coupled maps, we also study a real system of two parametrically forced pendulums with an unidirectional coupling, and find the same bicritical scaling behavior. In addition, this kind of bicritical behavior was also found both in an electronic system of two periodically driven nonlinear LC-circuits with an unidirectional coupling [4] and in a system of two unidirectionally-coupled Chua's circuits [6]. It is thus believed that the bicriticality in the abstract system of the unidirectionally coupled 1D maps may be observed in a real system consisting of two period-doubling subsystems with an unidirectional coupling.

Figure 1 shows the stability diagram of periodic orbits in the unidirectionally coupled 1D maps for $C = 0.45$. The numbers inside the different regions denote the period of the oscillation in the second subsystem. Stability of an orbit with period q is determined by its stability multipliers,

$$\lambda_1 = \prod_{t=1}^{q}(-2Ax_t), \quad \lambda_2 = \prod_{t=1}^{q}(-2By_t). \tag{3}$$

Here λ_1 and λ_2 determine the stability of the first and second subsystems, respectively. As the parameter A is increased, the first subsystem exhibits a sequence of period-doubling bifurcations at the vertical straight lines, where $\lambda_1 = -1$, accumulating at a critical line, denoted by a vertical dashed line. When crossing the vertical critical line, a transition to chaos occurs in the first subsystem. For small values of the parameter B, the period of oscillation in the second subsystem is the same as that in the first subsystem, as in the case of forced oscillation. As B is increased for a fixed value of A, the second subsystem also undergoes a sequence of period-doubling bifurcations at the non-vertical lines where $\lambda_2 = -1$, accumulating at a critical line, denoted by a non-vertical dashed line. When crossing the non-vertical critical line, a transition to chaos takes place in the second subsystem. Note that these two critical lines meet at a bicritical point, denoted by a solid circle, corresponding to a border of chaos in both subsystems.

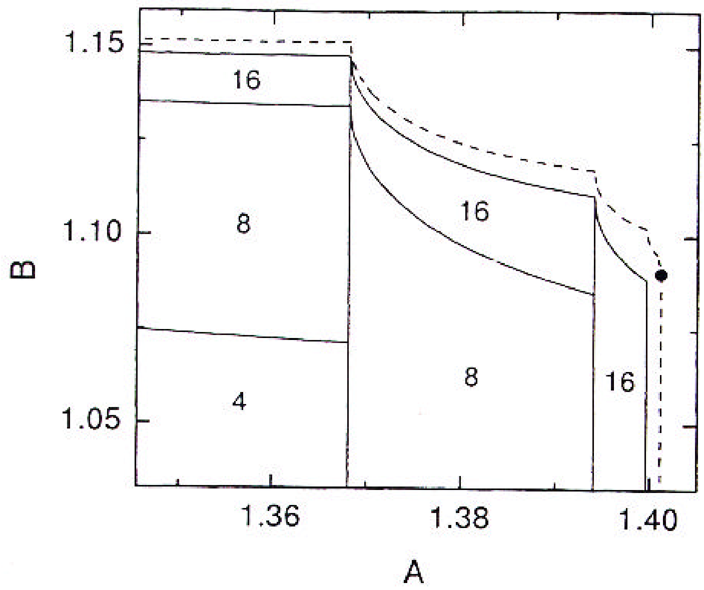

FIGURE 1. Stability diagram of the periodic orbits born via period-doubling bifurcations in the unidirectionally coupled 1D maps for $C = 0.45$. The numbers in the different regions represent the period of motion in the second subsystem. The vertical and non-vertical dashed lines denote the critical lines for the first and second subsystems, respectively. These two critical lines meet at a bicritical point, denoted by a solid circle, corresponding to a border of chaos in both subsystems. For other details, see the text.

To locate the bicritical point with a satisfactory precision, we numerically follow the orbits of period $q = 2^n$ up to level $n = 21$ in a quadruple precision, and obtain the "self-similar" sequences of both the parameters (A_n, B_n) converging to the bicritical point and the orbit points (x_n, y_n) approaching the origin. We first note that the sequences of A_n and x_n in the first subsystem are the same as those in the 1D maps. Hence the scaling behavior in the first subsystem becomes the same as that in the 1D maps [1]. That is, the sequences $\{A_n\}$ and $\{x_n\}$ accumulate to their limit values, $A = A_c$ $(= 1.401\,155\,189\,092\cdots)$ and $x = 0$, geometrically as follows:

$$A_n - A_c \sim \delta_1^{-n}, \quad x_n \sim \alpha_1^{-n} \text{ for large } n. \tag{4}$$

The scaling factors δ_1 and α_1 are just the Feigenbaum constants δ $(= 4.669\cdots)$ and α $(= -2.502\cdots)$ for the 1D maps, respectively. However, the second subsystem exhibits a non-Feigenbaum critical behavior, unlike the case of the first subsystem. The two sequences $\{B_n\}$ and $\{y_n\}$ also converge geometrically to their limit values $B = B_c$ $(= 1.090\,094\,348\,701)$ and $y = 0$, respectively:

$$B_n - B_c \sim \delta_2^{-n}, \quad y_n \sim \alpha_2^{-n} \text{ for large } n. \tag{5}$$

Here the two scaling factors δ_2 and α_2 are given by

$$\delta_2 \simeq 2.3928, \quad \alpha_2 \simeq -1.5053. \tag{6}$$

Note that these scaling factors are completely different from those in the first subsystem (i.e., the Feigenbaum constants for the 1D maps). For more details on the scaling results obtained by a direct numerical method, refer to Ref. [7].

We now employ the eigenvalue-matching method [8] and numerically make the RG analysis of the bicritical behavior in the unidirectionally-coupled map T of Eq. (2). The basic idea is to associate a value (A', B') for each (A, B) such that $T^{(n+1)}_{(A',B')}$ locally resembles $T^{(n)}_{(A,B)}$, where $T^{(n)}$ is the 2^nth-iterated map of T (i.e., $T^{(n)} = T^{2^n}$). A simple way to implement this idea is to linearize the maps in the neighborhood of their respective fixed points and equate the corresponding eigenvalues.

Let $\{z_t\}$ and $\{z'_t\}$ be two successive cycles of period 2^n and 2^{n+1}, respectively, i.e.,

$$z_t = T^{(n)}_{(A,B)}(z_t), \quad z'_t = T^{(n+1)}_{(A',B')}(z'_t); \; z_t = (x_t, y_t). \tag{7}$$

Here x_t depends only on A, but y_t is dependent on both A and B, i.e., $x_t = x_t(A)$ and $y_t = y_t(A, B)$. Then their linearized maps at z_t and z'_t are given by

$$DT^{(n)}_{(A,B)} = \prod_{t=1}^{2^n} DT_{(A,B)}(z_t), \quad DT^{(n+1)}_{(A',B')} = \prod_{t=1}^{2^{n+1}} DT_{(A',B')}(z'_t). \tag{8}$$

(Here DT is the linearized map of T.) Let their eigenvalues, called the stability multipliers, be $(\lambda_{1,n}(A), \lambda_{2,n}(A,B))$ and $(\lambda_{1,n+1}(A'), \lambda_{2,n+1}(A', B'))$. The recurrence relations for the old and new parameters are then given by equating the stability multipliers of level n, $\lambda_{1,n}(A)$ and $\lambda_{2,n}(A, B)$, to those of the next level $n+1$, $\lambda_{1,n+1}(A')$ and $\lambda_{2,n+1}(A', B')$, i.e.,

$$\lambda_{1,n}(A) = \lambda_{1,n+1}(A'), \quad \lambda_{2,n}(A, B) = \lambda_{2,n+1}(A', B'). \tag{9}$$

The fixed point (A^*, B^*) of the renormalization transformation (9), gives the bicritical point (A_c, B_c). By linearizing the renormalization transformation (9) at the fixed point (A^*, B^*), we have a linearized matrix Δ_n

$$\begin{pmatrix} \Delta A \\ \Delta B \end{pmatrix} = \Delta_n \begin{pmatrix} \Delta A' \\ \Delta B' \end{pmatrix}, \tag{10}$$

where $\Delta A = A - A^*$, $\Delta B = B - B^*$, $\Delta A' = A' - A^*$, $\Delta B' = B' - B^*$. After some algebra, we obtain the analytic formulas for the eigenvalues $\delta_{1,n}$ and $\delta_{2,n}$ of the matrix Δ_n,

$$\delta_{1,n} = \frac{\left.\frac{d\lambda_{1,n+1}}{dA'}\right|_*}{\left.\frac{d\lambda_{1,n}}{dA}\right|_*}, \quad \delta_{2,n} = \frac{\left.\frac{\partial\lambda_{2,n+1}}{\partial B'}\right|_*}{\left.\frac{\partial\lambda_{2,n}}{\partial B}\right|_*}. \tag{11}$$

Here the asterisk denotes the fixed point (A^*, B^*). As $n \to \infty$, $\delta_{1,n}$ and $\delta_{2,n}$ approach δ_1 and δ_2, which are just the parameter scaling factors in the first and second subsystems, respectively. Note also that as in the 1D case, the local rescaling factors of the state variables are simply given by

$$\alpha_{1,n} = \left.\frac{dx}{dx'}\right|_*, \quad \alpha_{2,n} = \left.\frac{dy}{dy'}\right|_*. \tag{12}$$

Here $\alpha_{1,n}$ and $\alpha_{2,n}$ also converge to the orbital scaling factors, α_1 and α_2, in the first and second subsystems, respectively.

With increasing the level up to $n = 15$, we first solve the fixed-point equation of the renormalization transformation (9) and obtain the bicritical point,

$$(A^*, B^*) = (1.401\,155\,189\,092\,050\,6, 1.090\,094\,348\,701). \tag{13}$$

Next, we use the formulas of Eqs. (11) and (12) and obtain the parameter and orbital scaling factors,

$$\delta_1 = 4.669\,201\,609\,1, \quad \delta_2 = 2.392\,73, \tag{14}$$

$$\alpha_1 = -2.502\,907\,874\,8, \quad \alpha_2 = -1.505\,31. \tag{15}$$

Note that these RG results agree well with those obtained by a direct numerical method. For more details on the RG results, refer to Ref. [7].

In order to confirm the above bicritical scaling behavior, we also study a real system of two parametrically-forced pendulums with a one-way coupling. Its dynamics is governed by the equations,

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = f_A(x_1, y_1, t), \tag{16a}$$

$$\dot{x}_2 = y_2 + C(x_2 - x_1), \quad \dot{y}_2 = f_B(x_2, y_2, t) + C(y_2 - y_1), \tag{16b}$$

where $f_A(x, \dot{x}, t) = -2\pi\beta\Omega\dot{x} - 2\pi(\Omega^2 - A\cos 2\pi t)\sin 2\pi x$. Using both the direct numerical method and the eigenvalue-matching RG method, we investigate the scaling behavior near a bicritical point (A_c, B_c) $[= (0.798\,049\,182\,451\,9, 0.802\,377\,2)]$ for $\beta = 1.0$, $\Omega = 0.5$ and $C = -0.2$. The bicritical scaling behavior is thus found to be the same as that in the unidirectionally coupled 1D maps [9].

As an evidence of scaling, we present a simple example for the case of unidirectionally coupled pendulums. Figure 2 shows the attractors for the three values of (A, B) near the bicritical point (A_c, B_c). All these attractors are the hyperchaotic ones with two positive Lyapunov exponents [10],

$$\sigma_1 = \lim_{m\to\infty} \frac{1}{m} \sum_{t=1}^{m} \ln|2Ax_t|, \quad \sigma_2 = \lim_{m\to\infty} \frac{1}{m} \sum_{t=1}^{m} \ln|2By_t|. \tag{17}$$

Here the first and second Lyapunov exponents σ_1 and σ_2 denote the average exponential divergence rates of nearby orbits in the the first and second subsystems,

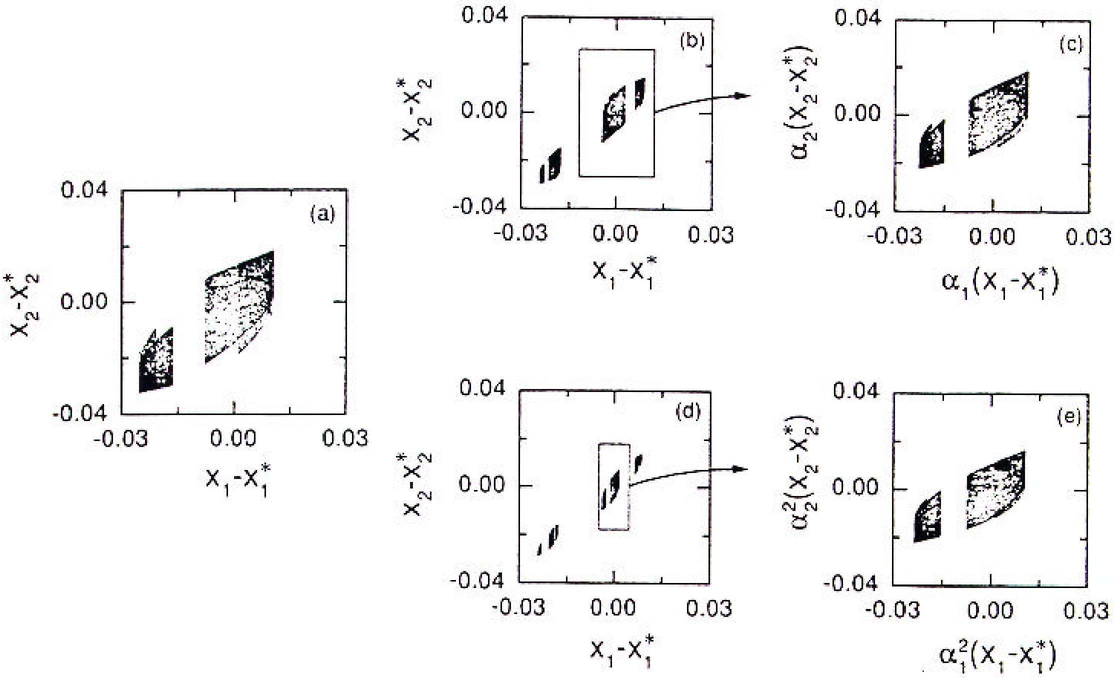

FIGURE 2. Hyperchaotic attractors near the point (x_1^*, x_2^*) [= (0.100 545 53, 0.100 111)] in the unidirectionally coupled pendulums for the three values of (A, B) near the bicritical point (A_c, B_c); in (a) $(A, B) = (A_c + \Delta A, B_c + \Delta B)$ ($\Delta A = 0.00085$, $\Delta B = 0.0037$), in (b) and (c) $(A, B) = (A_c + \Delta A/\delta_1, B_c + \Delta B/\delta_2)$, and in (d) and (e) $(A, B) = (A_c + \Delta A/\delta_1^2, B_c + \Delta B/\delta_2^2)$. For other details, see the text.

respectively. Figure 2(a) shows the hyperchaotic attractor with $\sigma_1 \simeq 0.107$ and $\sigma_2 \simeq 0.045$ around the point (x_1^*, x_2^*) [= (0.100 545 53, 0.100 111)] in the $x_1 - x_2$ plane for $A = A_c + \Delta A$ and $B = B_c + \Delta B$, where $\Delta A = 0.00085$ and $\Delta B = 0.0037$. To see scaling, we first rescale ΔA and ΔB with the parameter scaling factors δ_1 and δ_2, respectively. The attractor for the rescaled parameter values of $A = A_c + \Delta A/\delta_1$ and $B = B_c + \Delta B/\delta_2$ is shown in Fig. 2(b). It is also the hyperchaotic attractor with $\sigma_1 \simeq 0.055$ and $\sigma_2 \simeq 0.023$. We next magnify the region in the small box by the scaling factor α_1 for the x_1 axis and α_2 for the x_2 axis, and then we get the picture in Fig. 2(c). Note that the picture in Fig. 2(c) reproduces the previous one in Fig. 2(a) approximately. Repeating the above procedure once more, we obtain the two pictures in Figs. 2(d) and 2(e). That is, Fig. 2(d) shows the hyperchaotic attractor with $\sigma_1 \simeq 0.027$ and $\sigma_2 \simeq 0.012$ for $A = A_c + \Delta A/\delta_1^2$ and $B = B_c + \Delta B/\delta_2^2$. Magnifying the region in the small box with the scaling factors α_1^2 for the x_1-axis and α_2^2 for the x_2-axis, we also obtain the picture in Fig. 2(e), which reproduces the previous one in Fig. 2(c) with an increased accuracy. The details on the bicritical behavior in the unidirectionally coupled pendulums will be given elsewhere [9].

To sum up, we have studied the bicritical behavior of period doublings in unidirectionally coupled 1D maps by using both the direct numerical method and the eigenvalue-matching RG method. It has been thus found that for the bicritical case, a new type of non-Feigenbaum critical behavior appears in the second (response) subsystem, while the first (drive) subsystem is in the Feigenbaum critical state. We have also confirmed this kind of bicritical behavior in a system of parametrically forced pendulums with an unidirectional coupling. It is thus believed that the bicriticality in the abstract system of unidirectionally-coupled 1D maps may be observed in a real system consisting of two period-doubling subsystems with a one-way coupling.

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