Bifurcations in 2D Area-Preserving Mappings

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1. Symmetric Structures of the Periodic Orbits

We consider a class of two-dimensional area-preserving mappings of

$$T : x_{n+1} = 2h(x_n) - y_n \ ; \ y_{n+1} = x_n. \tag{1}$$

The area-preserving property of T is represented by det M = 1, where $M = \prod_{i=0}^{n-1}$ with

$$M_i = \begin{pmatrix} 2h'(x_i), & -1 \\ 1, & 0 \end{pmatrix}$$

and this property is satisfied for any function h(x). The residue R is a convenient quantity in discussing the stability of an orbit [1] and it is given as

$$R = (2 - TrM)/4.$$

Also useful is the concept of reversibility which has greatly facilitated numerical studies of period doubling sequences. When T is reversible, the symmetry S of T exists and T = (TS)S with $S^2 = 1 = (TS)^2$ so that the inverse of T is given by $T^{-1} = STS$. The S and TS are complementary symmetries and they are

$$S \ : \ x_{t+1} = y_t \ ; \ y_{t+1} = x_t$$

$$TS \ : \ x_{t+1} = 2h(y_t) - x_t \ ; \ y_{t+1} = y_t.$$

The S reflects the point with respect to the symmetry line y = x while TS with respect to the symmetry curve x = h(y) keeping y constant. When h(x) is an antisymmetric function h(-x) = -h(x) another pair of symmetries that can not be derived from the previous ones exists. They are

$$S \ : \ x_{t+1} = -x_t \ ; \ y_{t+1} = y_t - 2h(x_t)$$

$$TS \ : \ x_{t+1} = -y_t \ ; \ y_{t+1} = -x_t$$

and the symmetry lines are now y axis (x = 0) and y = -x. These pairs are not unique because T^nS and TS T^{-n} for any integer n can be complementary pairs of T.

Symmetries are very useful in locating periodic orbits. An orbit on a symmetry curve with even numbered period can be found from the condition that the midpoint of the orbit is on the same symmetry curve. An orbit with odd numbered period on the S curve can be found from the condition that the (n+1)/2 way point lies on the TS curve. The structure of periodic orbits is obtained by operating T and T^{-1} on the

$$x_{\frac{n}{2}+\ell} = x_{\frac{n}{2}-\ell-1} \quad ; \quad y_{\frac{n}{2}+\ell+1} = y_{\frac{n}{2}-\ell}$$

$$(\ell = 0,\ 1,\ 2,\ \dots,\ \tfrac{n}{2} - 1)$$

for the even numbered periodic orbit on S,

$$x_{\frac{n}{2}+\ell} = x_{\frac{n}{2}-\ell-2} \quad ; \quad y_{\frac{n}{2}+\ell+1} = y_{\frac{n}{2}-\ell-1}$$

$$(\ell = 0,\ 1,\ \dots,\ \tfrac{n}{2} - 2)$$

for the even numbered periodic orbit on TS, while

$$x_{\frac{n+1}{2}+\ell-1} = x_{\frac{n+1}{2}-\ell-1} \quad ; \quad y_{\frac{n+1}{2}+\ell} \quad y_{\frac{n+1}{2}-\ell}$$

$$(\ell = 0,\ 1,\ \dots,\ \tfrac{n-1}{2})$$

for the odd numbered periodic orbit with the (n+1)/2 way point on TS and the initial point on S.

2. Equivalence of Various 2D Area Preserving Mappings

It is shown in ref. 2 that various 2D area-preserving mappings with one parameter quadratic nonlinearity are equivalent. When we choose for $h(x)$

$$h_{HL}(x) = cx + x^2$$

the mapping of (1) becomes equivalent to Helleman's mapping while the quadratic mappings of DeVogelarere and Henon are obtained with

$$h_{DV}(x) = px - (1-p)x^2$$

$$h_{HN}(x) = \frac{1}{2}(1 - ax^2).$$

From these relations the equivalent parameters of one mapping can be obtained from those of the other mapping. For example, the Feigenbaum sequence of the bifurcation parameters p_n and the n-period fixed points $P_n{}^* = (x_n{}^*, y_n{}^*)$ of DeVogelarere mapping are obtained from those of Hellman's mapping $\{C_n\}$ and $\{P_n{}^{*HL}\}$

$$P_n = C_n$$

$$P_n{}^{*DV} = -\frac{1}{1-C_n} P_n{}^{*HL}.$$

Similarly Helleman's parameters are related to Henon's by

$$C_n{}' = 1 \pm \sqrt{1 + a_n}$$

and

$$P_n{}^{*HL} = -\frac{a_n}{2} P_n{}^{*HN} - \frac{C_n}{2}.$$

form is the standard mapping of Chirikov given by

$$T_c : r_{n+1} = r_n - \mu \sin 2\pi \theta_n \; ; \; \theta_{n+1} = \theta_n + r_{n+1}. \tag{3}$$

The mapping T_c and T of (1) are equivalent if we choose for h(x)

$$h(x) = x - \frac{\mu}{2} \sin 2\pi x. \tag{4}$$

The linear transformation of $T \to T_c$ is

$$Q_c : x = \theta \; ; \; y = \theta - r \tag{5}$$

and its inverse is given by

$$Q_c^{-1} : \theta = x : r = x - y. \tag{6}$$

Any result obtained in one mapping can be translated into the other one by these transformations. For example, from the n-periodic orbit (x_n^*, y_n^*) of T, the n-periodic orbit (r_n^*, θ_n^*) of T_c is easily found to be $(r_n^* = x_n^* - y_n^*)$. Two symmetries S and TS of (2) are given as

$$S_c \; : \; r_{t+1} = -r_t \; ; \; \theta_{t+1} = \theta_t - r_t$$
$$T_cS_c \; : \; r_{t+1} = -r_t + \mu \sin 2\pi (r_t - \theta_t) \; ; \; \theta_{t+1} = \theta_t + \mu \sin 2\pi (r_t - \theta_t) - 2r_t, \tag{7}$$

with $S_c^2 = 1 = (T_cS_c)^2$. Therefore the symmetry lines of $y = x$ and $x = h(y)$ in the Chirikov's coordinate become respectively

$$r = 0 \text{ and } r = \frac{\mu}{2} \sin 2\pi (r - \theta). \tag{8}$$

The generalized Devogelarere mapping [1] of

$$T_D : X_D' = -Y_D + f(X_D) \; ; \; Y_D' = X_D - f(X_D')$$

and T of (1) are shown to be equivalent by the transformations Q_D and Q_D^{-1} of

$$Q_D \; : \; x = X_D \; ; \; y = Y_D + f(X_D)$$
$$Q_D^{-1} \; : \; X_D = x \; ; \; Y_D = y - f(x)$$

with $f(x) = h(x)$. GREENE et al. [1] show that two symmetries $S1_D$ and $S2_D$ of

$$S1_D \; : \; X_D' = X_D \; ; \; Y_D' = -Y_D$$
$$S2_D \; : \; X_D' = Y_D + f(X_D) \; ; \; Y_D' = X_D - X_D'$$

are convenient because the (dominant) symmetry $S1_D$ is the X_D axis making the analysis of the period doubling bifurcation sequence rather simple.

3. Period-Doubling Bifurcations

In the following a sequence of period-doubling bifurcation is described. A mapping T with a parameter a has a stable interval. As the parameter a changes the stable orbit becomes hyperbolically unstable with reflection and new orbits orbit on its symmetry line bifurcates on the same symmetry line. The initial point $P_0 = (x_0, y_0)$ of a 2n-period orbit can be found from the solution of the simultaneous equations $x_0 = h(y_0)$ and $x_n(p_0) = h(y_n(P_0))$. If we define $g_n(z) = y_n(P_0)$, with $x_0 = h(z)$ and $y_0 = z$, then some roots of the equation

$$\phi_n(z) \equiv g_n(z) - z = 0 \tag{9}$$

will give the initial point of the n-period orbit on the symmetry line. When we construct a function $\psi_{2n}(z)$ from $\phi_n(z)$ as

$$\psi_{2n}(z) = \phi_n(\phi_n(z) + z) + \phi_n(z) \tag{10}$$

then we can show [3] that some of the roots of the equation $\psi_{2n}(z) = 0$ give the initial point of the 2n period. We assume that the period-doubling bifurcation occurs at $a = a_n$. As a approaches a_n from below (assuming the stable interval to be $a < a_n$) both $\phi_n(y\,;\, a < a_n)$ and $\psi_{2n}(y\,;\, a < a_n)$ cross the y axis at $y_0 = y_n^*$ ($a < a_n$) of the initial point of the n-period orbit with the negative slopes, since an n-period orbit is also a 2n-period orbit. As a increases the negative slopes, $\phi'(y_n^*\,;\, a < a_n)$, become steeper until $\phi'(y_n^*\,;\, a = a_n) = -2$ while $|\psi'_{2n}(y_n^*\,;\, a)| \to 0$ as $a \to a_n$. This can be seen by differentiating (9) as

$$\psi'_{2n}(z) = \phi'_n(\phi_n(z) + z)\ (\phi'_n(z) + 1) + \phi'_n(z)$$

and putting $z = y_n^*$ to get

$$\psi'_{2n}(y_n^*) = \phi'_n(y_n^*)(\phi'_n(y_n^*) + 2).$$

Thus the condition of the bifurcation of the n-period orbit at $a = a_n$ is that $\phi'_n(y_n^*\,;\, a_n) + 2 = 0$ and $\phi'_n(y_n^*\,;\, a \lesseqgtr a_n) + 2 \lesseqgtr 0$ where ϕ_n is defined in (9).

We can also show [3] that for the mapping T of (1) we have

$$\phi'_n(y_n^*\,;\, a) = -2R. \tag{11}$$

From the observations of GREENE et al. [1], we see then that as R increases from $R < 1$ to $R > 1$ corresponding to $a < a_n$ and $a > a_n$ the stable elliptical orbit ($0 < R < 1$) becomes unstable at $R = 1$ ($a = a_n$) and it changes into a hyperbolic orbit with reflection ($R > 1$) and thus a period-doubling bifurcation occurs. An odd periodic orbit usually does the period-doubling bifurcation on the S-symmetry line, in which case the above sequence of period-doubling bifurcation is reestablished, but replacing $\phi_n(z)$ in $\psi_{2n}(z)$ of (10) by $\xi_n(z)$ defined by

$$\xi_n(z) = \frac{1}{2}\{x_n(z) + y_n(z)\} - z$$

where $(x_n(z), y_n(z)) = T^n P(z,z)$, an n iterate of T, a point on the S-symmetry line $P(z,z)$. Then we can show [4] that

$$\xi_n'(x_0\,:\, a) = -2R$$

similar to (11).

Making use of (3)-(8) we have studied numerically [5] a sequence of period-doubling bifurcations of periodic orbits of Chirikov mapping. We followed one sequence on a symmetry line which leads to an accumulation point of the parameter value $\mu_c = 0.8259739 \ldots$. The sequence also confirms the universal scaling behavior and universal constants of $\delta = 8.721097 \ldots$, $\alpha = -4.0180767 \ldots$ and $\beta = 16.363896 \ldots$, where the Feigenbaum ratios are defined as

$$\delta = \lim_{k\to\infty} [(a_{k-2}-a_{k-1})/(a_{k-1}-a_k)]$$

$$\alpha = \lim_{k\to\infty} (d_{k-1}/d_k), \quad \beta = \lim_{k\to\infty} (e_{k-1}/e_k).$$

Here a_k denotes the parameter value at which 2^{k+1} periodic orbits are born and d_k and e_k are distances between bifurcated points on and off the symmetry line respectively. We are also studying the period-trebling cascades of $k \cdot 3^n$. Preliminary results show that the universal constant values are $\delta = 430$, $\alpha = -44$ and $\beta = -187$.

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