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On the Consequence of Blow-Out Bifurcations

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We investigate the consequence of a blow-out bifurcation of a chaotic attractor in an invariant line in a family of piecewise linear planar maps by changing a positive parameter β controlling the reinjection. Through a supercritical blow-out bifurcation, a chaotic or hyperchaotic attractor, exhibiting on-off intermittency, is born, depending on the value of β. For large β, a hyperchaotic attractor with a positive second Lyapunov exponent appears. However, as the parameter β decreases and passes a threshold value β^*, a transition from hyperchaos to chaos occurs. Hence, for $0 < \beta < \beta^*$ a chaotic attractor with a negative second Lyapunov exponent is born. The sign of the second Lyapunov exponent of the newly-born intermittent attractor is found to be determined through competition between its laminar and bursting components. When the "strength" (*i.e.*, a weighted second Lyapunov exponent) of the bursting component is larger (smaller) than that of the laminar component, a hyperchaotic (chaotic) attractor appears.

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I. INTRODUCTION

Many dynamical systems of interest possess an invariant subspace of the whole phase space and exhibit interesting dynamical behavior. For example, this situation occurs naturally in the synchronization of chaotic oscillators [1–4] and in systems with spatial symmetries [5]. Particularly, the phenomenon of chaos synchronization has attracted much attention because of its potential practical applications (*e.g.*, see Ref. 6).

An important problem in the field of chaos synchronization concerns the stability of a chaotic attractor in the invariant subspace S [7,8]. When the Lyapunov exponents corresponding to perturbations transverse to S are all negative, the chaotic state in S is stable and is an attractor in the whole phase space. However, as a coupling parameter passes a threshold value, the chaotic attractor can become transversely unstable (*i.e.*, its largest transverse Lyapunov exponent becomes positive) via a blow-out bifurcation [5,9–11]. Depending on the global dynamics, two kinds of blow-out bifurcations may occur. For the case of a supercritical (or soft) blow-out bifurcation, a new attractor appears and exhibits intermittent bursting, called on-off intermittency [12–20]; long periods of motion near S (off state) are occasionally interrupted by short-term burstings away from S (on state). On the other hand, for the case of a subcritical (or hard) blow-out bifurcation, an abrupt disappearance of the chaotic state occurs, and typical trajectories starting near S are attracted to another distant asynchronous attractor (or infinity).

Here, we are interested in the type of intermittent attractors born via supercritical blow-out bifurcations. In particular, we are interested in whether the intermittent bursting attractor born at the blow-out bifurcation is hyperchaotic (*i.e.*, has more than one positive Lyapunov exponent) or not. Examples of both hyperchaotic attractors [9,21,22] and chaotic attractors (*i.e.*, an attractor with only one positive Lyapunov exponent) [9,23] were given in previous works. However, the dynamical origin for the appearance of such hyperchaotic and chaotic bursting attractors remains unclear.

In this paper, we study the consequence of blow-out bifurcations in a family of piecewise linear planar maps by changing a parameter β (> 0) controlling the reinjection. For large β, a hyperchaotic attractor is born through a supercritical blow-out bifurcation. However, for a value of β smaller than a threshold value β^*, a chaotic attractor appears. In Sec. II, we investigate the mechanism for the transition from hyperchaos to chaos. A typical trajectory on the newly-born attractor, exhibiting on-off intermittency, may be decomposed into laminar (*i.e.*, nearly constant) and bursting components. The type of the intermittent attractor is found to be determined through competition between its laminar and bursting components. When the "strength" (*i.e.*, its weighted second Lyapunov exponent) of the bursting component is larger (smaller) than that of the laminar component, an intermittent hyperchaotic (chaotic) attractor appears.

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Finally, a summary is given in Sec. III.

II. TYPE OF INTERMITTENT ATTRACTORS BORN VIA BLOW-OUT BIFURCATIONS

We investigate the type of intermittent attractors born through blow-out bifurcations in a three-parameter family of piecewise linear planar mappings of $[0, 1] \times \mathrm{R}$ to itself [9]:

$$T : x_{n+1} = f(x_n),\ y_{n+1} = g(x_n, y_n); \tag{1}$$

$$f(x) = \begin{cases} x/\alpha & \text{for } 0 \leq x \leq \alpha, \\ (x-\alpha)/(1-\alpha) & \text{for } \alpha < x \leq 1, \end{cases} \tag{2}$$

$$g(x, y) = \begin{cases} \gamma y & \text{for } y < 1 \text{ and } 0 \leq x \leq \alpha, \\ y/\gamma & \text{for } y < 1 \text{ and } \alpha < x \leq 1, \\ 1 + \beta(1-y) & \text{for } y \geq 1, \end{cases} \tag{3}$$

where x_n and y_n are state variables of the driving and the response subsystems at a discrete time n, respectively, and the parameters are $\alpha \in (0, 1)$, $\gamma > 0$, and $\beta \in \mathrm{R}$. We also assume a symmetry $y \to -y$ to define the map in the lower half plane.

We note that the map T has an invariant line $y = 0$, on which a chaotic attractor exists. The longitudinal stability of trajectories on the chaotic attractor against a perturbation along the invariant line $y = 0$ is determined by its longitudinal Lyapunov exponent,

$$\sigma_{||} = -\alpha \ln \alpha - (1-\alpha)\ln(1-\alpha), \tag{4}$$

which is just the Lyapunov exponent in the uncoupled 1D map f. On the other hand, the transverse stability of the chaotic attractor against a perturbation across the invariant line $y = 0$ (*i.e.*, a perturbation along the y-axis) is determined by its transverse Lyapunov exponent,

$$\sigma_{\perp} = \varepsilon \ln \gamma,\ \ \varepsilon = 2\alpha - 1. \tag{5}$$

Without loss of generality, we can assume that $\gamma > 1$. Then, for negative ε, the chaotic attractor on the invariant line $y = 0$ becomes transversely stable because its transverse Lyapunov exponent $\sigma_{\perp}$ is negative. However, as ε is increased and passes a threshold value ε^* $(= 0)$, the transverse Lyapunov exponent $\sigma_{\perp}$ becomes positive. Consequently, when passing ε^*, the chaotic attractor becomes transversely unstable; then, a supercritical (subcritical) blow-out bifurcation occurs for positive (negative) β [9]. Hereafter, we investigate the consequence of supercritical blow-out bifurcations by varying the parameter β (> 0) controlling the reinjection.

To determine the type of an attractor born through a supercritical blow-out bifurcation, its Lyapunov exponents are numerically calculated as follows. We choose a random initial orbit point with uniform probability in the range of $x \in (0, 1)$ on a line $y = \delta$ $(= 10^{-3})$ near

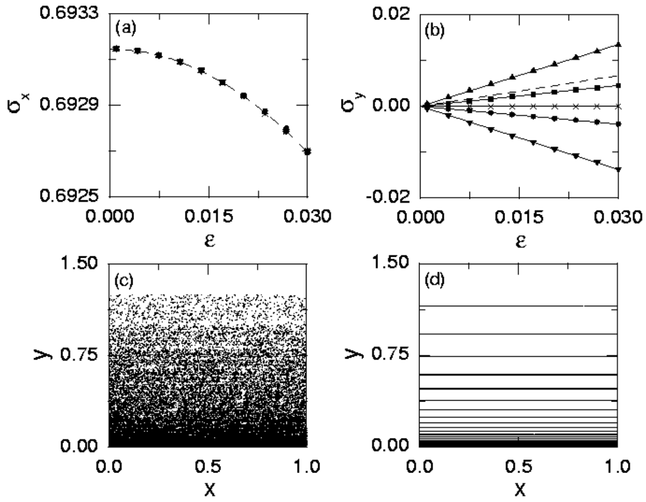

Fig. 1. Plots of the (a) first (σ_x) and the (b) second (σ_y) largest Lyapunov exponents of the intermittent attractors newly born through supercritical blow-out bifurcations versus ε for $\gamma = 1.25$ with $\beta = 1.4$ (up triangles), 0.9 (squares), 0.8 (crosses), 0.7 (circles), and 0.5 (down triangles). The length of a trajectory segment for the calculation of σ_x and σ_y is $L = 5\times10^7$. For reference, the longitudinal and the transverse Lyapunov exponents of the chaotic attractor on the invariant line $y = 0$, $\sigma_{||}$ and $\sigma_{\perp}$, are represented by dashed lines in (a) and (b), respectively. The data for σ_y are well fitted with the straight solid lines, and their slopes for $\beta = 1.4$, 0.9, 0.8, 0.7, and 0.5 are 0.45, 0.15, 0, −0.13, and −0.46, respectively. For $\varepsilon = 0.02$, examples of (c) hyperchaotic ($\sigma_x = 0.693$ and $\sigma_y = 0.009$) and (d) chaotic ($\sigma_x = 0.693$ and $\sigma_y = -0.0092$) attractors are given for $\beta = 1.4$ and 0.5, respectively. In both (c) and (d), $(x_0, y_0) = (0.5, 0.01)$, 5×10^3 points are computed before plotting, and the next 5×10^4 points are plotted.

the invariant line $y = 0$ and follow the trajectory until its length L becomes 5×10^7 [24]. Then the Lyapunov exponents, σ_x and σ_y, of the trajectory segment with length L against perturbations along the x and y directions are given by

$$\sigma_x = \frac{1}{L}\sum_{n=0}^{L-1} r_n^{(x)},\ \ r_n^{(x)} = \ln|f_x(x_n)|, \tag{6}$$

$$\sigma_y = \frac{1}{L}\sum_{n=0}^{L-1} r_n^{(y)},\ \ r_n^{(y)} = \ln|g_y(x_n, y_n)|, \tag{7}$$

where the subscript i $(i = x, y)$ in f and g denotes a partial derivative with respect to the variable i.

Figures 1(a) and 1(b) show σ_x and σ_y of the attractors born through blow-out bifurcations for $\beta = 1.4$ (up triangles), 0.9 (squares), 0.8 (crosses), 0.7 (circles), and 0.5 (down triangles). Note that the Lyapunov exponents σ_x and σ_y correspond to the first and the second largest Lyapunov exponents, respectively. (Hereafter, σ_x and σ_y will be referred to as the first and the second Lyapunov exponents, respectively.) Due to the skew-product structure of the map T, the first Lyapunov exponent σ_x is just the longitudinal Lyapunov exponent $\sigma_{||}$ of the

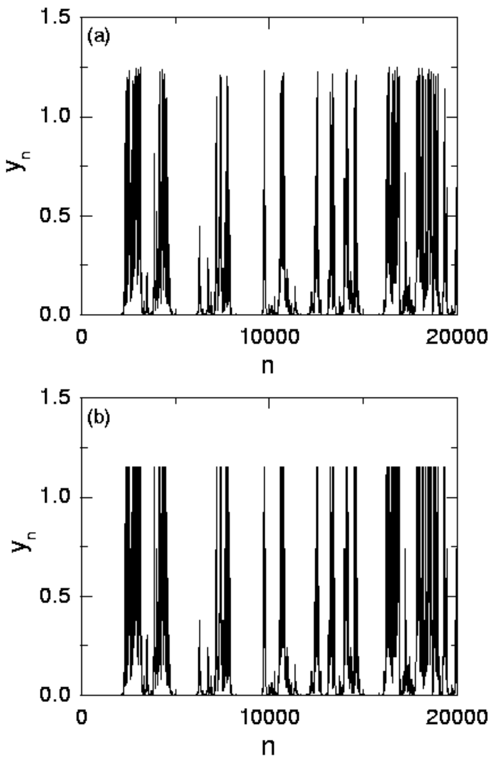

Fig. 2. Time series of the transverse variable y, representing the deviation from the $y = 0$ line, for $\gamma = 1.25$ and $\varepsilon = 0.02$ with (a) $\beta = 1.4$ and (b) $\beta = 0.5$. In both cases, the initial orbit point is $(x_0, y_0) = (0.5, 0.01)$.

chaotic attractor on the invariant line $y = 0$, independently of β. For this case, the type of the newly born attractor is determined through the sign of the second Lyapunov exponent σ_y. The data for σ_y are well fitted with straight lines; hence, they scale linearly with ε [9]. For $\beta = 1.4$, the attractor is hyperchaotic with $\sigma_y > 0$. On the other hand, as β is decreased the magnitude of the slope of σ_y decreases, eventually it becomes zero for a threshold value β^* (= 0.8), and then the slope becomes negative [see Fig. 1(b)]. Hence, a chaotic attractor with $\sigma_y < 0$ appears for $\beta < \beta^*$. As examples for $\varepsilon = 0.02$, Figs. 1(c) and 1(d) show the hyperchaotic ($\sigma_x = 0.693$ and $\sigma_y = 0.009$) and the chaotic ($\sigma_x = 0.693$ and $\sigma_y = -0.0092$) attractors when $\beta = 1.4$ and 0.5, respectively.

As shown in Fig. 2, the time series of the transverse variable y of typical trajectories on the newly born attractors exhibits on-off intermittency, in which long episodes of nearly constant evolution are occasionally interrupted by short-term bursts. To characterize the on-off intermittent time series, we use a small quantity y^* for the threshold value of y such that for $y < y^*$, the signal is considered to be in the laminar (off) state and for $y \geq y^*$, it is considered to be in the bursting (on) state. So far, the statistical properties of such on-off intermittent attractors have been well characterized through investigation of the distribution of the laminar lengths and the scaling of the average laminar length and the average bursting amplitude [14–20].

However, although examples were given in previous works (*e.g.*, see Refs. 9 and 21-23), the dynamical origin of the hyperchaotic and chaotic intermittent attractors through blow-out bifurcations remains unclear. Hence, we investigate the mechanism for the transition from hyperchaos to chaos by varying the parameter β. As explained above, a typical trajectory, exhibiting on-off intermittency, may be decomposed into its laminar and bursting components. Then, the second Lyapunov exponent σ_y of an attractor [see Eq. (7) for the second Lyapunov exponent σ_y of a trajectory segment] can be given by the sum of the two weighted second Lyapunov exponents of the laminar and the bursting components, Λ_y^l and Λ_y^b:

$$\sigma_y = \Lambda_y^l + \Lambda_y^b \quad (8)$$
$$= \Lambda_y^b - |\Lambda_y^l|, \quad (9)$$

where the laminar component always has a negative weighted second Lyapunov exponent ($\Lambda_y^l < 0$). Here, the weighted second Lyapunov exponent Λ_y^i for each component ($i = l, b$) is given by the product of the fraction, μ_i, of time spent in the i state and its second Lyapunov exponent σ_y^i, *i.e.*,

$$\Lambda_y^i = \mu_i \sigma_y^i; \quad \mu_i = \frac{L^i}{L},$$
$$\sigma_y^i = \frac{1}{L^i} \sum_{n \in i\,\mathrm{state}}{}' r_n^{(y)} \quad (i = l, b), \quad (10)$$

where L^i is the time spent in the i state for a trajectory segment of length L and the primed summation is performed in each i state. As can be seen in Eq. (9), the sign of σ_y is determined through competition between the laminar and the bursting components. Hence, when the "strength" (*i.e.*, the weighted second Lyapunov exponent Λ_y^b) of the bursting component is larger (smaller) than that (*i.e.*, $|\Lambda_y^l|$) of the laminar component, an intermittent hyperchaotic (chaotic) attractor appears. We also note that the weighted Lyapunov exponents Λ_y^l and Λ_y^b depend on the threshold value y^*, although σ_y is independent of y^*. With decreasing y^*, Λ_y^l decreases to zero because the time μ_l spent in the laminar state goes to zero; thus, Λ_y^b [$= |\Lambda_y^l| + \sigma_y$] converges to σ_y. Here, we again emphasize that σ_y, determining the type of intermittent attractors, depends only on the difference between Λ_y^b and $|\Lambda_y^l|$, which is independent of y^* [see Eq. (9)]. Hence, although $\Lambda_y^{l(b)}$ depends on y^*, the conclusion as to the type of intermittent attractors is independent of y^*. Hereafter, we fix the value of the threshold value of y at $y^* = 10^{-2}$.

Figures 3(a) and 3(b) show the weighted second Lyapunov exponents of the laminar and the bursting components, Λ_y^l and Λ_y^b, respectively. As mentioned above, the

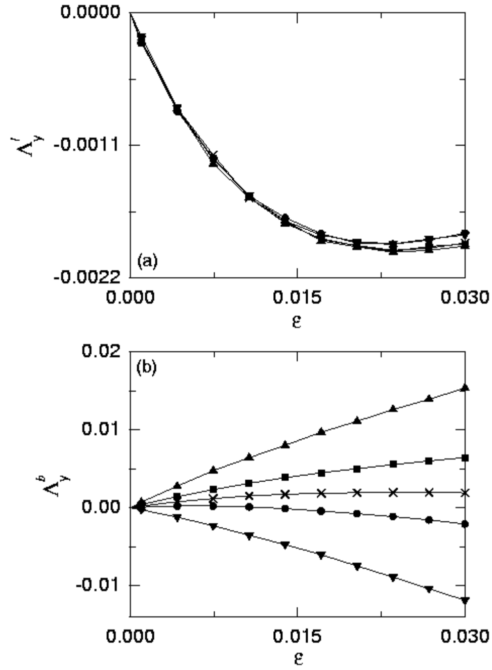

Fig. 3. Plots of the weighted second Lyapunov exponents Λ_y^l and Λ_y^b of the (a) laminar and the (b) bursting components versus ε for $\gamma = 1.25$ and $y^* = 10^{-2}$ with $\beta = 1.4$ (up triangles), 0.9 (squares), 0.8 (crosses), 0.7 (circles), and 0.5 (down triangles). Straight line segments between neighboring data symbols are plotted only to guide the eye. For small ε Λ_y^l is nearly the same, independently of β, while Λ_y^b decreases with decreasing β. Eventually, for $\beta = \beta^*$ $(= 0.8)$ $\Lambda_y^b = |\Lambda_y^l|$. Hence, for $\beta > \beta^*$ $\Lambda_y^b > |\Lambda_y^l|$, while for $\beta < \beta^*$ $\Lambda_y^b < |\Lambda_y^l|$.

type of newly born intermittent attractor is determined through competition between the laminar and the bursting components as follows. We first note that for small ε, Λ_y^l is nearly the same, independently of β. On the other hand, Λ_y^b decreases with decreasing β. Eventually, for a threshold value $\beta = \beta^*$ $(= 0.8)$, the strength of the laminar and bursting components becomes balanced (*i.e.*, $\Lambda_y^b = |\Lambda_y^l|$). Consequently, for $\beta > \beta^*$, there is a hyperchaotic attractor with $\sigma_y > 0$ because the bursting component is dominant (*i.e.*, $\Lambda_y^b > |\Lambda_y^l|$), while for $\beta < \beta^*$, there is a chaotic attractor with $\sigma_y < 0$ because the laminar component is dominant (*i.e.*, $\Lambda_y^b < |\Lambda_y^l|$).

The fraction $\mu_{l(b)}$ of the laminar (bursting) time (*i.e.*, the time spent in the laminar (bursting) state) and the second Lyapunov exponent $\sigma_y^{l(b)}$ of the laminar (bursting) component are also given in Fig. 4. For small ε both μ_l and σ_y^l are nearly independent of β, and hence the weighted second Lyapunov exponent Λ_y^l $(= \mu_l \sigma_y^l)$ becomes nearly the same, independently of β. On the other hand, σ_y^b decreases with decreasing β, although its fraction μ_b $(= 1 - \mu_l)$ of the bursting time is nearly independent of β. Consequently, the weighted second Lyapunov exponent Λ_y^b $(= \mu_b \sigma_y^b)$ decreases with decreasing β. Thus, for a threshold value β^*, $\Lambda_y^b = |\Lambda_y^l|$; then,

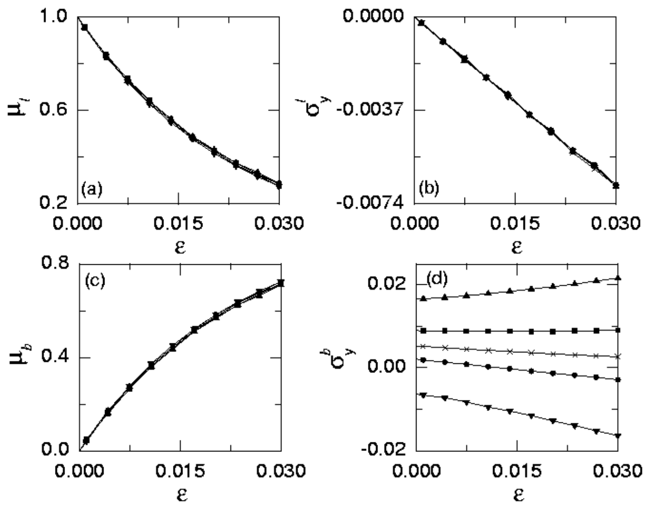

Fig. 4. Plots of (a) [(c)], the fraction $\mu_{l(b)}$ of the laminar (bursting) time, and (b) [(d)], the second Lyapunov exponent $\sigma_y^{l(b)}$ of the laminar (bursting) component, versus ε for $\gamma = 1.25$ and $y^* = 10^{-2}$ with $\beta = 1.4$ (up triangles), 0.9 (squares), 0.8 (crosses), 0.7 (circles), and 0.5 (down triangles). Straight line segments between neighboring data symbols are plotted only to guide the eye.

a transition from hyperchaos to chaos occurs.

We believe that the transition we have found from a hyperchaotic to a chaotic intermittent attractor can be understood as follows: After the blow-out bifurcation, the intermittent attractor includes an infinite number of unstable periodic orbits that are off the invariant line $y = 0$. Some of these unstable periodic orbits have two positive Lyapunov exponents, and some others have only one positive Lyapunov exponent. We conjecture that as β is decreased, the "strength" of the group of asynchronous unstable periodic orbits with negative second Lyapunov exponents might increase, which may result in the observed decrease in σ_y^b.

III. SUMMARY

We have investigated the type of intermittent attractors born via blow-out bifurcations in a family of piecewise linear planar maps by varying the parameter β controlling the reinjection. When β is larger than a threshold value β^*, an intermittent hyperchaotic attractor with a positive second Lyapunov exponent is born, while for $\beta < \beta^*$ an intermittent chaotic attractor with a negative second Lyapunov exponent appears. The type of a newly-born on-off intermittent attractor is found to be determined via competition between its laminar and bursting components. If the bursting (laminar) component becomes dominant, then an intermittent hyperchaotic (chaotic) attractor appears.

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[24] Due to finite numerical precision, the computer compiler regards very small (nonzero) numbers as zeros when their magnitudes are less than a threshold value r^* for numerical underflow ($r^* = 1.7 \times 10^{-308}$ for IEEE double precision). Once the magnitude of the transverse variable y becomes less than r^*, the computed trajectory falls into an exactly constant state ($y = 0$) and further bursting from the $y = 0$ line cannot occur. For this case, we choose another random initial orbit points and repeat the procedure for calcualting the Lyapunov exponents until a trajectory segment of length L ($= 5 \times 10^7$) is obtained.