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Destruction of Chaotic Attractors in Coupled Chaotic Systems

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We investigate the sudden destruction of hyperchaotic attractors in symmetrically coupled one-dimensional maps. An asynchronous hyperchaotic attractor may appear through a blow-out bifurcation, where the synchronous chaotic attractor on the invariant synchronization line becomes unstable with respect to perturbations transverse to the synchronization line. It is found that such a hyperchaotic attractor may be broken up suddenly through stabilization of a periodic saddle embedded in the hyperchaotic attractor via a reverse subcritical pitchfork or period-doubling bifurcation. After break up of the hyperchaotic attractor, the asymptotic state transforms from the hyperchaotic state to a periodic state. Note that this sudden destruction of the hyperchaotic attractor occurs without any collision with its basin boundary, in constrast to the boundary crisis and is robust under a small perturbation of parameter mismatching.

Recently, synchronization in coupled chaotic systems has attracted much attention, in particular, in connection with applications to secure communication [1]. For the case of chaos synchronization, a synchronous chaotic attractor (SCA) exists on an invariant synchronization subspace S [2]. As a coupling parameter passes through a threshold value, the SCA becomes unstable under a perturbation transverse to S, and then a new asynchronous hyperchaotic attractor (HCA) with more than one positive Lyapunov exponent [3] may appear. Here we are interested in the fate of such asynchronous HCAs with respect to the strength of a coupling parameter.

By varying the coupling parameter, we investigate the sudden break-up of HCAs in symmetrically coupled one-dimensional (1D) maps. It is found that, when the coupling parameter passes through a threshold value, a HCA breaks up suddenly when a periodic saddle embedded in the HCA becomes stabilized via a reverse subcritical pitchfork or period-doubling bifurcation. Then the asymptotic state of the system changes from a hyperchaotic state to a periodic state. Note also that the sudden destruction of the HCA occurs without any collision with its basin boundary. Hence the old basin becomes occupied by the new periodic attractor. This is in contrast to the case of the boundary crisis [4], where the sudden destruction of both a chaotic attractor and its basin occurs when the chaotic attractor collides with its basin boundary. In order to examine robustness of the destruction of the HCAs, we also introduce a small perturbation of parameter mismatching. It is thus found that the destruction phenomenon becomes robust under a small parameter mismatching, although the underlying bifurcation mechanism may be changed.

We consider a symmetrically coupled system T consisting of two identical 1D maps,

$$T : \begin{cases} x_{t+1} = 1 - a x_t^2 + c(y_t^2 - x_t^2), \\ y_{t+1} = 1 - a y_t^2 + c(x_t^2 - y_t^2), \end{cases} \qquad (1)$$

where x_t and y_t are state variables of the first and the second 1D maps at a discrete time t, a is the control parameter of the uncoupled 1D map, and c is a coupling parameter. The coupled map T is invariant under the exchange transformation $x \leftrightarrow y$, and hence it has an exchange symmetry. If an orbit lies on the invariant symmetry line, then it is called a synchronous orbit because $x_t{=}y_t$ for all t. Otherwise, it is called an asynchronous orbit. Hereafter, the invariant symmetry line will be referred to as the synchronization line.

Note that the coupled map T is noninvertible because its Jacobian determinant det(DT) becomes zero along the critical curves, $L_0 = \{(x,y) \in R^2 : x = 0 \text{ or } y = 0\}$. Segments of the critical curves of rank k, $L_k = T^k(L_0)$ $(k = 1, 2, \ldots)$, are used to bound a compact region of the phase space, inside which trajectories starting near the synchronization line are confined [5]. Such a region, controlling the global dynamics and acting as a trapping bounded vessel, is called an absorbing area. Furthermore, boundaries of an absorbing area can be obtained by the union of segments of the critical curves and portions of the unstable manifolds of unstable periodic orbits. For this case, the trapping region is called a mixed absorbing area.

The stability of the SCA with respect to a perturbation transverse to the synchronization diagonal is intimately associated with transverse bifurcations of periodic sad-

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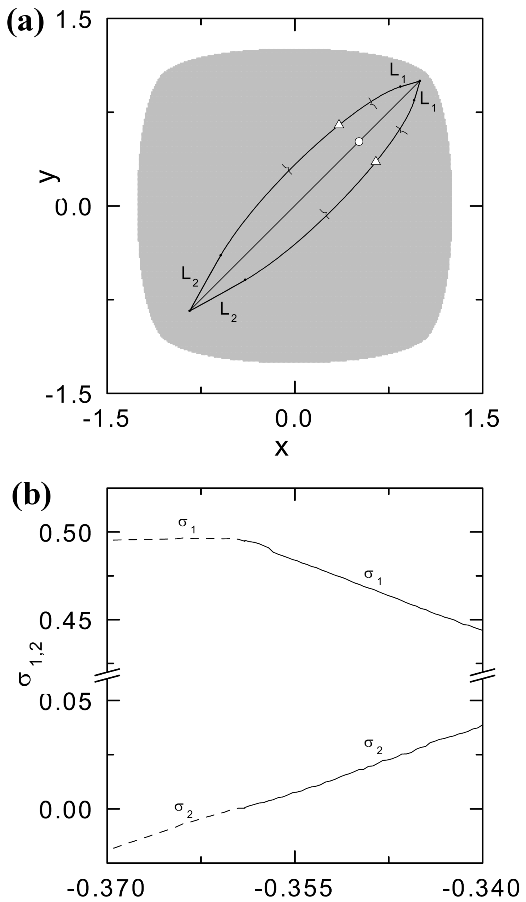

Fig. 1. (a) Weakly stable SCA on the synchronization diagonal for $a = 1.84$ and $c = -0.42$, which is surrounded by a mixed absorbing area, bounded by the union of segments of the unstable manifolds of the symmetric period-2 saddle, denoted by the up-triangles, and segments of the critical curves L_1 and L_2. The symmetric period-2 saddle is born from the period-1 orbit on the diagonal, denoted by the circle, through a supercritical period-doubling bifurcation, and the gray region denotes the basin of the SCA. For other details, see the text. (b) Lyapunov exponents near the blow-out bifurcation point $c_{b,r}$ ($\simeq -0.359$), where the transverse Lyapunov exponent of the SCA crosses zero. As a result of the blow-out bifurcation, a HCA with two positive Lyapunov exponents appears. Here the dashed (solid) lines denote the Lyapunov exponents of the SCA (HCA).

dles embedded in the SCA [6,7]. If all the periodic saddles are transversely stable, the SCA becomes asymptotically (or strongly) stable (Lyapunov stable and attracting in the usual topological sense). However, the SCA becomes weakly stable (*i.e.* Lyapunov unstable) in the Milnor sense [8] through a riddling bifurcation, where a first perodic saddle becomes transversely unstable. After the riddling bifurcation, some of the trajectories starting near the diagonal may be locally repelled. However, the fate of the locally repelled trajectories depends on the existence of an absorbing area [7,9]. When an absorbing

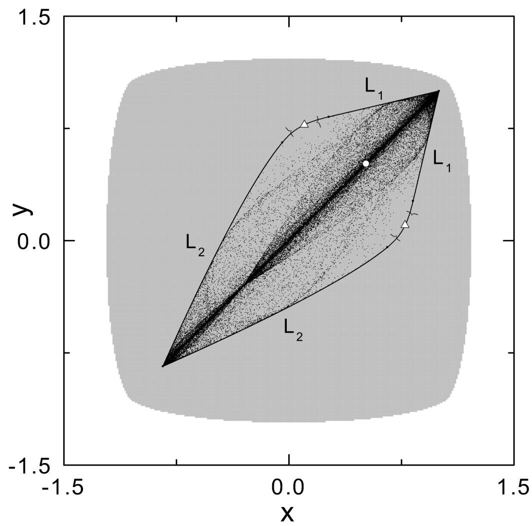

Fig. 2. HCA, born through a blow-out bifurcation, for $a = 1.84$ and $c = -0.35$.

area exists, the locally repelled trajectories are restricted to move inside the absorbing area, and are eventually attracted to the diagonal. With further change of the coupling parameter, a weakly-stable SCA loses its transverse stability through a blow-out bifurcation [10], where its transverse Lyapunov exponent becomes positive, and then it transforms to a chaotic saddle. The global effect of the blow-out bifurcation also depends on the existence of the absorbing area [7,9]. In the presence of an absorbing area, the blow-out bifurcation is gradual. Consequently, a new asynchronous HCA bounded to the absorbing area appears.

From now on, we present concrete examples of the sudden break-up of HCAs. For $a = 1.84$, a SCA with a single band exists on the synchronization diagonal. The SCA is strongly stable in the region of $c_{r,l}(= -1.406\,419\,...) \leq c \leq c_{r,r}(= -0.433\,579\,...)$ because all periodic orbits embedded in the SCA are transversely stable. When passing the right endpoint $c_{r,r}$, a riddling bifurcation occurs and then the strongly-stable SCA becomes weakly stable. For this case, the period-1 saddle embedded in the SCA becomes transversely unstable via a supercritical period-doubling bifurcation, leading to the birth of a new asynchronous period-2 saddle. Note also that the asynchronous saddle with period 2 is a symmetric one because it is invariant under the exchange operation $x \leftrightarrow y$. Figure 1(a) shows the weakly-stable SCA for $c = -0.42$ (after the riddling bifurcation). The gray region denotes the basin of the SCA. The segments of the unstable manifolds (whose directions are denoted by the arrows) of the symmetric period-2 saddle, denoted by the open up-triangle, connect with the segments of the critical curves L_1 and L_2 (the dots indicate where these segments connect) and, hence, define a mixed absorbing area, surrounding the SCA, in which the period-1 repeller, denoted by the open circle, is embedded. With a fur-

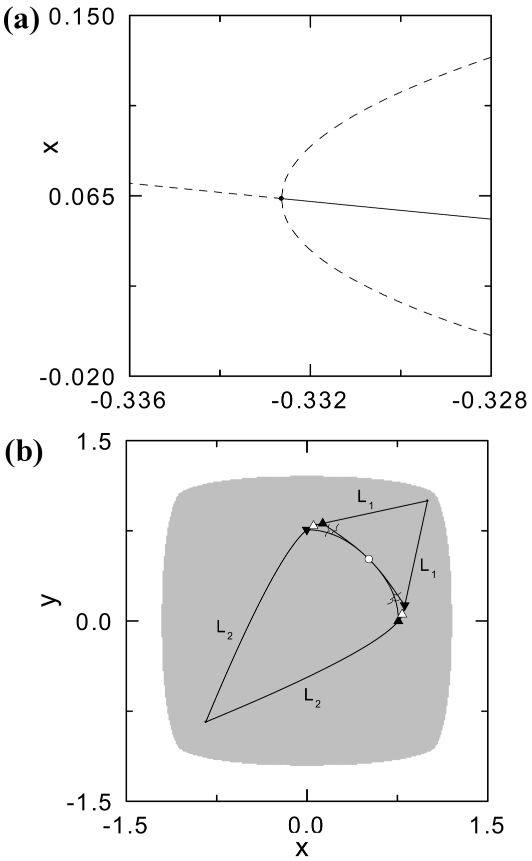

Fig. 3. (a) Bifurcation diagram (x versus c) for $a = 1.84$ showing that the symmetric period-2 saddle becomes stabilized by emitting a pair of asymmetric period-2 saddles through a subcritical pitchfork bifurcation. Here, unstable orbits are denoted by dashed lines while the stabilized orbit is denoted by the solid line. (b) Stabilized symmetric period-2 attractor, denoted by the open up-triangles, in the gray basin and a pair of asymmetric period-2 saddles, denoted by the solid up- and down-triangles for $a = 1.84$ and $c = -0.328$. All of these orbits lie inside an absorbing area bounded by segments of the critical curves L_1 and L_2.

ther increase in the coupling parameter, the SCA loses its transverse stability through a blow-out bifurcation for $c = c_{b,r}(\simeq -0.359)$, and then an asynchronous HCA bounded to the mixed absorbing area appears. Figure 1(b) shows the Lyapunov exponents near the blow-out bifurcation point $c_{b,r}$. The dashed (solid) lines denote the Lyapunov exponents σ_1 and σ_2 of the SCA (HCA). As c passes $c_{b,r}$, the transverse Lyapunov exponent σ_2 of the SCA crosses zero, and hence the SCA transforms to a chaotic saddle. Then, a new asynchronous HCA with two positive Lyapunov exponents developss from the synchronizaton diagonal. Figure 2 shows a HCA with $\sigma_1 \simeq 0.471$ and $\sigma_2 \simeq 0.018$ for $c = -0.35$.

With further change in the coupling parameter, the HCA destructs suddenly through the stabilization of the symmetric period-2 saddle embedded in the HCA for

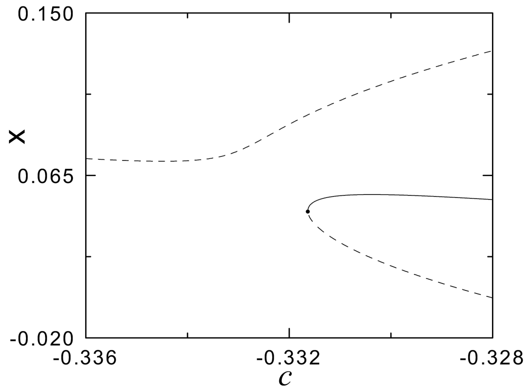

Fig. 4. Bifurcation diagram (x versus c) for $a = 1.84$ and $\epsilon = 0.001$. Here, the dashed lines denote unstable orbits while the solid line denotes a stable orbit.

$c = c_s$ $(= -0.332\,632\,...)$. Figure 3(a) shows the bifurcation diagram near the stabilization point c_s. Note that the symmetric period-2 saddle becomes stabilized by emitting a pair of asymmetric period-2 saddles through a reverse subcritical pitchfork bifurcation. Figure 3(b) shows the stabilized symmetric period-2 attractor, denoted by the open up-triangles, in the gray basin for $c = -0.328$. A pair of asymmetric period-2 saddles, denoted by solid up- and down-triangles, are also shown. All of them lie inside the absorbing area bounded by the critical curves L_1 and L_2. This stabilization of the symmetric period-2 attractor leads to the sudden destruction of the HCA, and then the asymptotic state of the system jumps from the hyperchaotic state to the stabilized symmetric period-2 state. Note that this destruction of the HCA occurs without any collision with its basin boundary and that the old basin is preserved to become the basin of the new symmetric period-2 attractor. This is in constrast to the case of the boundary crisis, where the sudden destruction of both a chaotic attractor and its basin takes place when the chaotic attractor collides with its basin.

We also consider the case where the coupling parameter c is decreased toward the left riddling bifurcation point $c_{r,l}$. As in the above case, a sudden destruction of the HCA occurs through stabilization of a periodic saddle embedded in the HCA, although the underlying bifurcation mechanisms are different. When passing the point $c_{r,l}$, the strongly stable SCA becomes weakly stable through a riddling bifurcation. For this case, the period-1 saddle embedded in the SCA becomes transversely unstable via a supercritical pitchfork bifurcation, leading to the birth of a pair of asynchronous asymmetric period-1 saddles. With a further decrease in c, the weakly stable SCA also loses its transverse stability via a blow-out bifurcation, and then a HCA with two positive Lyapunov exponents appears. However, a sudden destruction of this HCA occurs when the asymmetric period-1 saddles embedded in the HCA become stabilized through reverse subcritical period-doubling bifurcations. Then the asymptotic state of the system transforms from the hyperchaotic state to one of two asymmetric period-1 states, depending on the initial conditions.

In order to examine the robustness of the destruction of the HCAs, we introduce a small parameter mismatching by taking the control parameter b of the second 1D map as $b = a - \epsilon$, where a is the control parameter of the first 1D map and ϵ is a small symmetry-breaking parameter. It is thus found that the destruction of a HCA occurs through appearance of a new periodic attractor in the HCA. Hence, the destruction phenomenon becomes robust under a small parameter mismatching. However, when symmetry-breaking pitchfork bifurcations take part in the process of destruction, the underlying bifurcation mechanism for the destruction may change. An example for $\epsilon = 0.001$ is shown in Fig. 4. As shown in the bifurcation diagram, the HCA breaks up suddenly when a stable periodic attractor appears for $c \simeq -0.332$ via a saddle-node bifurcation. Through a comparision of the bifurcation diagram in Fig. 3(a) with that in Fig. 4, one can easily see that the reverse subcritical pitchfork bifurcation for $\epsilon = 0$ is replaced by a saddle-node bifurcation for a non-zero ϵ.

To sum up, we have investigated the destruction of HCAs in symmetrically coupled 1D maps. It is thus found that a sudden destruction of the HCA occurs when a periodic saddle embedded in the HCA becomes stabilized via a reverse subcritical pitchfork or period-doubling bifurcation. After break up of the HCA, the asymptotic state of the system changes from the hyperchaotic state to a periodic state. It is also found that the destruction phenomenon is robust under a small parameter mismatching, although the underlying bifurcation mechanism may change in some cases. Note that this kind of destruction is observed in other coupled dynamical systems such as the coupled Hénon maps [11]. Hence, the sudden destruction of a HCA through stabilization of a periodic saddle embedded in the HCA seems to be a generic phenonemon occurring in coupled chaotic systems.

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