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Riddling Transition in Asymmetric Dynamical Systems

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We investigate the bifurcation mechanism for the loss of transverse stability of the chaotic attractor on an invariant subspace in the asymmetric dynamical system. It is found that a direct global-riddling transition occurs through a transcritical contact bifurcation between a periodic saddle embedded in the chaotic attractor on the invariant subspace and a repeller on its basin boundary. At the moment of this new type of riddling bifurcation, a bounded trapping region, called an absorbing area, disappears. Consequently, the basin becomes globally riddled with a dense set of repelling tongues leading to divergent orbits. This riddled basin is also characterized by the uncertainty exponents, and thus typical power-law scaling is found.

Many dynamical systems of interest possess an invariant subspace S in the whole phase space and exhibit interesting dynamical behaviors. For example, this situation occurs naturally in the synchronization of chaotic oscillators [1] and in the systems with spatial symmetries [2]. In particular, the phenomenon of chaos synchronization has attracted much attraction, because of its potential practical application in secure communication [3]. An important question in this field concerns stability of the chaotic attractor on S with respect to the perturbation transverse to S [4]. Such transverse stability of the chaotic attractor is intimately associated with transverse bifurcations of periodic saddles embedded in the chaotic attractor [5–8]. Interesting phenomena such as the intermittent bursting [9], riddled basins of attraction [10,11], and on-off intermittency [12] have been observed during the loss of transverse stability of the chaotic attractor.

The chaotic attractor on S is asymptotically (or strongly) stable (*i.e.*, Lyapunov stable and attracting in the usual topological sense) if all periodic saddles embedded in the chaotic attractor are transversely stable. However, it becomes weakly stable (*i.e.*, Lyapunov unstable) in the Milnor sense [13] through a riddling bifurcation, in which the first periodic saddle embedded in the chaotic attractor loses its transverse stability. After the riddling bifurcation, a dense set of locally repelling "tongues" opens from the transversely unstable repeller and its preimages. Hence the trajectories falling in the tongues will be repelled from S. However, the fate of the locally repelled trajectories depends on the global dynamics of the system [7,8]. In case of the supercritical riddling bifurcation, they are restricted by the nonlinear mechanism to move within an absorbing area, acting as a bounded trapping vessel, that lies strictly inside the basin, and exhibit transient intermittent bursting from S [9]. For this case the basin is said to be locally riddled. However, when the riddling bifurcation is subcritical, the nonlinear mechanism is too weak to restrict the motion to an absorbing area, and hence the locally repelled trajectories will go to another attractor (or infinity). Consequently the basin becomes globally riddled with a dense set of repelling tongues, belonging to the basin of another attractor (or infinity) [10,11].

Eventually, the chaotic attractor (with locally or globally riddled basin) on S loses its transverse stability through a blow-out bifurcation [14], where its transverse Lyapunov exponent becomes positive, and then it transforms to a chaotic saddle. The global effect of the blow-out bifurcation also depends on the existence of an absorbing area [7,8]. In the presence of the absorbing area, the blow-out bifurcation becomes gradual. Hence, a new chaotic attractor, bounded to the absorbing area, appears through a supercritical blow-out bifurcation, and it exhibits the on-off intermittency [12]. However, without the absorbing area, the chaotic attractor abruptly disappears through a subcritical blow-out bifurcation, and then typical trajectories starting near S are attracted to another distant attractor (or infinity).

In this paper, we investigate the mechanism for the loss of transverse stability of the chaotic attractor on S in the asymmetric dynamical systems. It is found that

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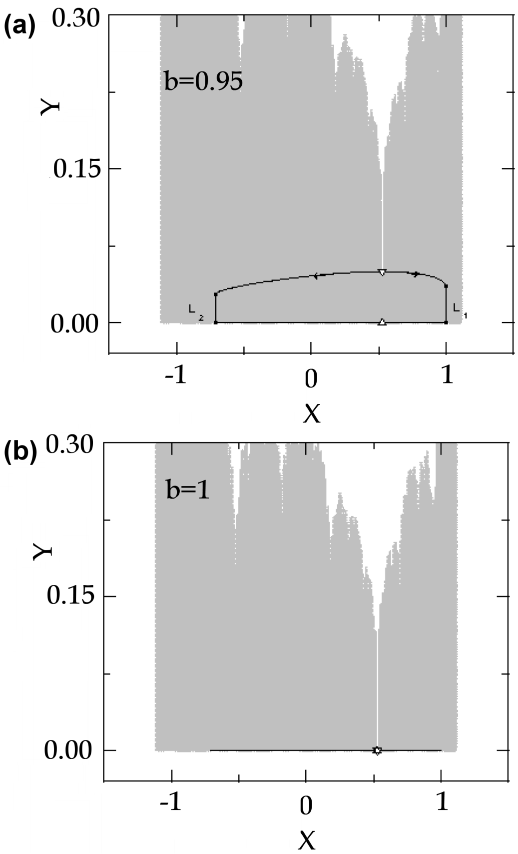

Fig. 1. Change in the structure of the basin (gray region) of the chaotic attractor on the $y = 0$ line for $a = 1.71$. (a) Union of segments of the unstable manifolds of the repeller ($\bigtriangledown$) at the basin boundary and segments of the critical curves L_1 and L_2 defines a mixed absorbing area of the chaotic attractor at $y = 0$ for $b = 0.95$. As b increases, the repeller approaches the saddle point ($\bigtriangleup$) embedded in the chaotic attractor at $y = 0$, and hence the absorbing area shrinks. (b) Through a transcritical bifurcation of the saddle point for $b = b_r$ $(= 1)$, the absorbing area disappears, and then the basin becomes globally riddled with a dense of tongues, leading to divergent trajectories. For other detail, see the text.

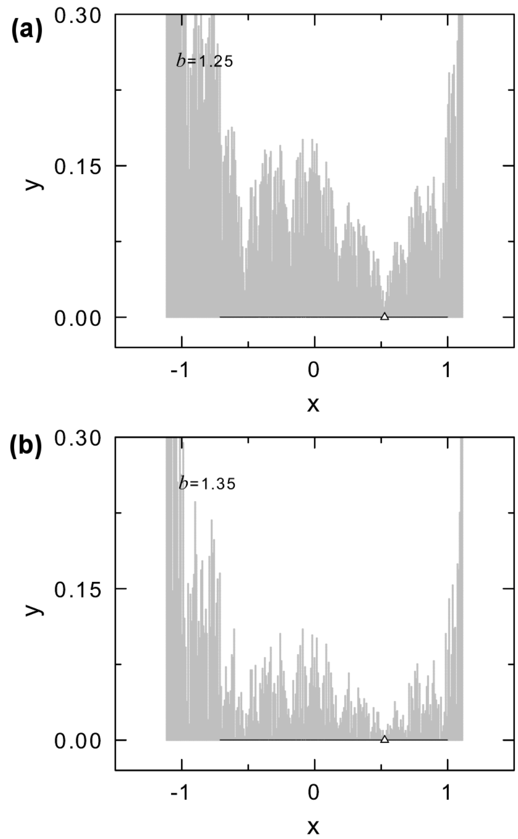

Fig. 2. Globally-riddled basins (gray region) of the chaotic attractor on the $y = 0$ line for (a) $b = 1.25$ and (b) $b = 1.35$. As b increases toward the blow-out bifurcation point $b \simeq 1.411$, the measure of the riddled basin decreases to zero.

a direct global-riddling transition takes place via a transcritical contact bifurcation between a periodic saddle embedded in the chaotic attractor on S and a repeller on the basin boundary. This bifurcation mechanism is in contrast to that in the symmetric dynamical systems, where the basin becomes globally riddled through a subcritical pitchfork [5] or period-doubling bifurcation [7]. After the global-riddling transition, the basin becomes a "fat fractal" [15], riddled with a dense set of tongues leading to divergent trajectories. The fine scaled riddling of the basin is characterized by the uncertainty exponents [11], and thus typical power-law scaling is found.

We consider a unidirectionally-coupled map T without symmetry,

$$T : \begin{cases} x_{t+1} = f(x_t) = 1 - a x_t^2, \\ y_{t+1} = b e^{-(x_t - x^*)^2} y_t + y_t^2. \end{cases} \tag{1}$$

where x_t and y_t are state variables of the first and second subsystems at a discrete time t, and a is the control parameter of the one-dimensional (1D) map $f(x)$. This coupled map T has an invariant line $y = 0$, in which the dynamics is described by the 1D map $f(x)$. The transverse stability of the attractor at $y = 0$ is governed by the factor $be^{-(x-x^*)^2}$. Here $b(> 0)$ is the coupling parameter, and the Gaussian function $e^{-(x-x^*)^2}$ becomes maximum at the fixed point x^* $[= (-1 + \sqrt{1 + 4a})/2a]$ of the 1D map $f(x)$. Hence the period-1 orbit $(x^*, 0)$ of the map T becomes the first periodic orbit to become transversely unstable. For the case of this unidirectional asymmetric coupling, the lowest order nonlinearity in y is y^2, because there is no symmetry. This is in contrast to the symmetric-coupling case with the exchange symmetry where the lowest order nonlinearity is y^3. Note that for any initial points $y_0 \geq 0$, trajectories have $y_t \geq 0$ for all subsequent times $(t > 0)$. Hence we consider only the dynamics in the upper half phase plane with $y \geq 0$.

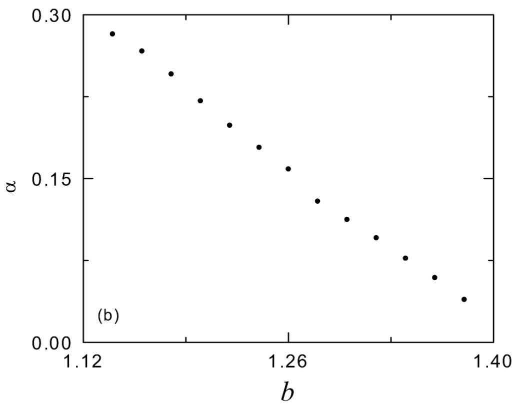

Fig. 3. Plot of the uncertainty exponent α versus b for $a = 1.71$.

We also note that the coupled map T is noninvertible, and its Jacobian determinant det(DT) becomes zero on the critical curves, $L_0 = \{(x, y) \in R^2 : x = 0 \text{ or } y = -\frac{b}{2}e^{-(x-x^*)^2}\}$. A finite number of segments of images $L_k[= T^k(L_0)\ (k = 1, 2, \ldots)]$ of the critical curves L_0 can be used to define the boundary of a compact absorbing area $\mathcal{A}$ with the properties that (i) $\mathcal{A}$ is trapping (*i.e.*, trajectories that enter $\mathcal{A}$ cannot leave it again), and (ii) $\mathcal{A}$ is superattracting (*i.e.*, every point sufficiently close to the boundary of $\mathcal{A}$ will jump into $\mathcal{A}$ after a finite number of iterations) [16]. Furthermore, boundaries of an absorbing area $\mathcal{A}$ can be also obtained by the union of segments of the critical curves L_k and segments of the unstable manifolds of unstable periodic orbits. For this case, $\mathcal{A}$ is called a mixed absorbing area.

As the control parameter a is increased, the coupled map T exhibits an infinite sequence of period-doubling bifurcations of attractors with period 2^n $(n = 0, 1, 2, \ldots)$ on the invariant $y = 0$ line, accumulating at a finite point a_∞ $(= 1.401\,155 \cdots)$. Beyond the accumulation point a_∞, chaotic attractors exist on the $y = 0$ line for the a values in a positive measure set, riddled with a dense set of periodic windows [15]. As an example, we consider the case of $a = 1.71$, where a chaotic attractor with a single band exists on the $y = 0$ line. This chaotic attractor becomes strongly stable, if all the periodic saddles embedded in it are transversely stable. However, it becomes weakly stable through a riddling bifurcation, where the the saddle fixed point, embedded in the chaotic attractor, loses its transverse stability via a transcritical bifurcation. We now discuss the effect of the transcritical riddling bifurcation on the chaotic attractor at $y = 0$ by increasing the coupling parameter b. As b increases toward the riddling bifurcation point $b_r(= 1)$, the structure of the basin changes, as shown in Fig. 1. For $b = 0.95$, the chaotic attractor is strongly stable, because all periodic saddles embedded in the chaotic attractor are stable. The basin for this case is denoted by the gray region in Fig. 1(a). The segments of the unstable manifolds (whose directions are denoted by the arrows) of the repeller, denoted by the down-triangle ($\bigtriangledown$), at the cusp of the basin boundary connect with the segments of the critical curves L_1 and L_2 (the dots indicate where these segments connect), and hence define a mixed absorbing area, surrounding the chaotic attractor at $y = 0$, in which the saddle, denoted by the up-triangle ($\bigtriangleup$), is embedded. As b is increased, the repeller approaches the saddle, and also the absorbing area shrinks. Eventually, at the riddling bifurcation point $b = b_r$, the repeller and saddle collide, and hence the absorbing area disappears [see Fig. 1(b)]. Note that the chaotic attractor at $y = 0$ is contacting its basin boundary at the saddle point. Consequently, an infinitely narrow "tongue," belonging to the basin of an attractor at the infinity, emanates at the saddle point. In fact, the whole basin becomes globally riddled with a dense set of repelling tongues, emanating from the saddle point and its preimages. When passing the point b_r, the repeller moves down off the basin boundary, and exchanges stability with the saddle [*i.e.*, the repeller (saddle) transforms to a saddle (repeller)]. However, the chaotic attractor continues to contact its basin boundary at a new repelling fixed point ($\bigtriangleup$). This is just the transcritical bifurcation, occurring in dynamical systems with some constraint (not a symmetry), when a Floquet multiplier passes 1 [17]. In such a way, a direct global-riddling transition occurs through a transcritical contact bifurcation between the saddle fixed point, embedded in the chaotic attractor and the repeller at the basin boundary. This new bifurcation mechanism is in contrast to that in the symmetric coupled systems [5,7].

With further increasing b toward the blow-out bifurcation point $b_b \simeq 1.411$, the repelling tongues, leading to divergent trajectories, continuously expands, as shown in Figs. 2(a) and 2(b). Hence the measure of the riddled basin of the chaotic attractor at $y = 0$ decreases to zero. At last, when b passes through the blow-out bifurcation point b_b, the chaotic attractor at $y = 0$ becomes transversely unstable, and then it transforms to a chaotic saddle.

We also characterize the fine scaled riddling of the basin by the uncertainty exponent α [11] with increasing b from 1.14 to 1.38. For a given b, consider a horizontal line $y = d$ $(d = 0.03)$. We choose an initial point at random with uniform probability in the range of $x \in (1, 1 - a)$ on the $y = d$ line. Also choose a second point z' at random within a distance 2ϵ of the first point z on the same line $y = d$. Then determine the final states of the trajectories starting with the two initial conditions z and z'. If the final states are different, the initial point z is said to be uncertain. We repeat this process for a large number of randomly chosen initial conditions until 2000 uncertain initial conditions are obtained, and estimate the probability $P(\epsilon)$ that the two initial conditions z and z' yield different final states. With decreasing ϵ, $P(\epsilon)$ exhibits a power-law scaling, $P(\epsilon) \sim \epsilon^\alpha$, where α is referred to as the uncertainty exponent. Note that, if $\alpha < 1$, then a substantial improvement in the accuracy of the initial conditions yields only a small decrease in the uncertainty of the final state. Figure 3 shows the plot of α versus b. As b increases toward the blow-out bifurcation point b_b, the value of α becomes smaller, and hence the uncertainty in determining the final state increases.

To sum up, we have investigated the loss of transverse stability of the chaotic attractor on the invariant space S in the asymmetric dynamical system, and found a new mechanism for the global-riddling transition through a transcritical contact bifurcation between a periodic saddle embedded in the chaotic attractor on S and a repeller on its basin boundary. Note that this bifurcation mechanism is in contrast to that in the symmetric dynamical systems. As a result of the riddling bifurcation, the basin becomes globally riddled with a dense set of tongues, leading to the divergent trajectories. Finally, this riddled basin has been also characterized by the uncertainty exponents, and thus typical power-law scaling has been found.

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