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Symmetry-Conserving and -Breaking Blow-Out Bifurcations in Coupled Chaotic Systems

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We consider blow-out bifurcations of synchronous chaotic attractors on invariant subspaces in coupled chaotic systems with symmetries. Through a blow-out bifurcation, the synchronous chaotic attractor becomes unstable with respect to perturbations transverse to the invariant subspace, and then a new asynchronous chaotic attractor may appear. However, the system symmetry may be preserved or violated when such a transition from synchronous to asynchronous chaotic motion occurs. Here we investigate the underlying mechanism for the symmetry preservation and violation. It is thus found that the shape of a minimal invariant absorbing area controlling the global dynamics and acting as a trapping bounded vessel determines whether the symmetry is conserved or not. For the case of a symmetric absorbing area, a symmetry-conserving blow-out bifurcation occurs while in the case of an asymmetric absorbing area, a symmetry-breaking blow-out bifurcation takes place.

In recent years, much attention has been paid to the phenomenon of synchronization in coupled chaotic systems. In the case of symmetrically coupled identical chaotic systems, such chaos synchronization may occur when a chaotic attractor (CA) on the symmetric invariant subspace S becomes stable with respect to perturbations transverse to S [1]. Hence, an important question concerns the transverse stability of the synchronous CA with regard to the coupling strength [2]. It is known that, as a coupling parameter passes a threshold value, the synchronous CA loses its transverse stability through a blow-out bifurcation [3], and then a new asynchronous CA appears in the supercritical (nonhysteric) case. However, the system symmetry may be conserved or broken when such a transition from synchronous to asynchronous motion occurs [4].

Here we investigate the underlying mechanism for the conservation and the breakdown of symmetry when a supercritical blow-out bifurcation takes place. Our new finding is that the shape of a minimal invariant absorbing area [5,6], which is a compact trapping region in the phase space, determines the type of blow-out bifurcation (*i.e.*, whether it is symmetry conserving or breaking). The shape of the minimal invariant absorbing area can be ascertained by the introduction of a small parameter mismatch between the subsystems [6]. In the case of a symmetric absorbing area, a symmetric asynchronous CA appears through a symmetry-conserving blow-out bifurcation. On the other hand, for the case of an asymmetric absorbing area, a symmetry-breaking blow-out bifurcation occurs, leading to the birth of a conjugate pair of asymmetric asynchronous CAs.

Let us consider the symmetrically coupled system T, consisting of two identical one-dimensional (1D) maps,

$$T : \begin{cases} x_{t+1} = 1 - a x_t^2 + g(x_t, y_t), \\ y_{t+1} = 1 - a y_t^2 + g(y_t, x_t), \end{cases} \quad (1)$$

where x_t and y_t are state variables of the first and second 1D maps at a discrete time t, a is the control parameter of the uncoupled 1D map, and g is a coupling function obeying the condition

$$g(x, x) = 0 \ \text{ for any } x. \quad (2)$$

This coupled map T has an exchange symmetry because it is invariant under the exchange of coordinates $x \leftrightarrow y$. The set of points which are invariant under the exchange operation forms an invariant symmetry line $y = x$. If an orbit lies on the invariant line, it is called a synchronous orbit because the two state variables x_t and y_t become the same for all t; otherwise, it is called an asynchronous orbit. Hereafter, the invariant line will be referred to as the synchronization line.

We also note that the coupled map T is non-invertible because its Jacobian determinant det(DT) (DT is the Jacobian matrix of T) becomes zero along the critical curves, $L_0 = \{(x, y) \in R^2 : [2ax - g_1(x, y)][2ay - g_1(y, x)] + g_2(x, y) g_2(y, x) = 0\}$, where the subscript i of g denotes the partial derivative of g with respect to the ith argument. Critical curves of rank k, L_k ($k = 1, 2, \ldots$), are then given by the images of rank k of L_0 [*i.e.*, $L_k = T^k(L_0)$]. Segments of these critical curves can be used to bound a compact region of the phase space that

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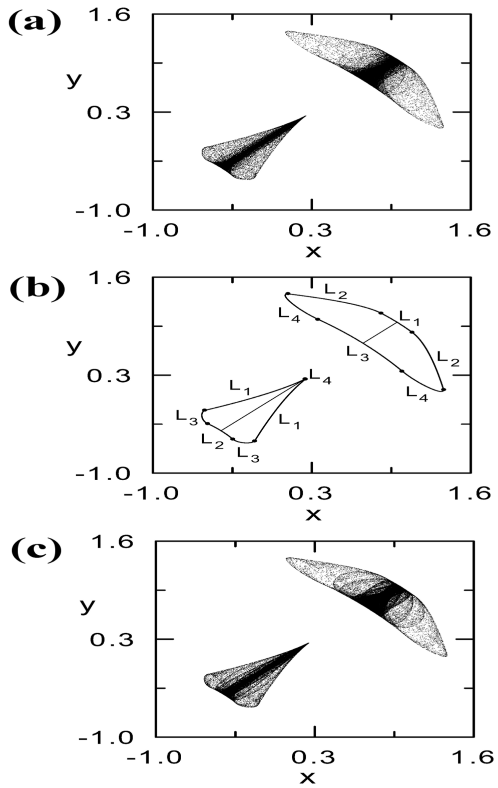

Fig. 1. Symmetry-conserving blow-out bifurcation for $a = 1.44$ in the linear-coupling case. (a) After the parameter mismatching with $\epsilon = 0.001$ for $c = -1.028$, a typical trajectory starting on the diagonal covers the entire minimal invariant absorbing area. (b) The minimal invariant absorbing area bounded by segments of the critical curves L_k ($k = 1, 2, 3, 4$) for $\epsilon = 0$. (c) A symmetric asynchronous CA for $c = -1.024$, born via a symmetry-conserving blow-out bifurcation.

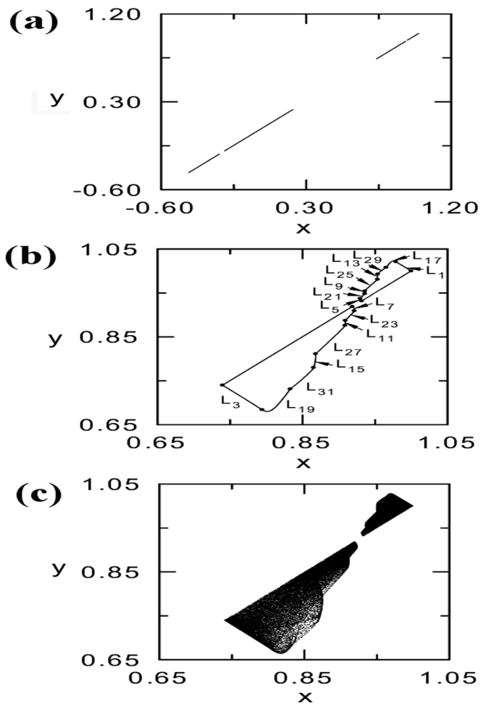

Fig. 2. Symmetry-breaking blow-out bifurcation for $a = 1.427$ in the linear-coupling case. (a) A synchronous CA on the diagonal with four bands for $c = -1.031$. (b) The minimal invariant absorbing area bounded by segments of the critical curves L_k ($k = 1, ..., 32$). For clear presentation, only a part of the entire absorbing area, including the upper two chaotic bands, is shown. (c) An asymmetric CA for $c = -1.027$, born via a symmetry-breaking blow-out bifurcation.

acts as a trapping bounded vessel, called an absorbing area $\mathcal{A}$, inside which trajectories starting near the synchronization diagonal are confined [7]. Furthermore, the boundaries of an absorbing area can also be obtained by the union of segments of critical curves and portions of unstable manifolds of unstable periodic orbits. For this case, $\mathcal{A}$ is called a mixed absorbing area.

Only the minimal invariant absorbing area (*i.e.*, the smallest invariant one including the synchronous CA) is important to characterize the global effect of the blow-out bifurcation. To ascertain the existence and shape of a minimal invariant absobing area near a blow-out bifurcation point, we introduce a small parameter mismatching by taking the control parameter b of the second 1D map as $b = a + \epsilon$, where a is just the control parameter of the first 1D map and ϵ is a small symmetry-breaking parameter. A consequence of such parameter mismatching is that invariance of the synchronization diagonal is lost. However, the existence of a minimal invariant absorbing area is generally persistent under a small parameter mismatching. Consequently, a typical trajectory starting on the diagonal covers the whole minimal invariant absorbing area, which gives us clear information on the shape of the minimal invariant absorbing area.

From now on, we present concrete examples of symmetry-conserving and -breaking blow-out bifurcations for a linear-coupling case with $g(x, y) = c(y - x)$, where c is a coupling parameter. For $a = 1.44$, a synchronous chaotic state with two bands exists on the synchronization diagonal. This synchronous chaotic state becomes an attractor for $c_{b,l}(\simeq -1.478) \leq c \leq c_{b,r}(\simeq -1.026)$ because its transverse Lyapunov exponent given by

$$\sigma_\perp = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \ln |2ax_t + 2c| \tag{3}$$

is negative. However, when passing the point $c_{b,r}$, a supercritical blow-out bifurcation occurs. To determine the type of this blow-out bifurcation, we introduce a small parameter mismatching with $\epsilon = 0.001$ for $c = -1.028$ (just before the blow-out bifurcation). For this mismatched case, the synchronization diagonal is no longer invariant, and a typical trajectory starting on the diagonal covers the whole minimal invariant absorbing area, as shown in Fig. 1(a). Based on this information given

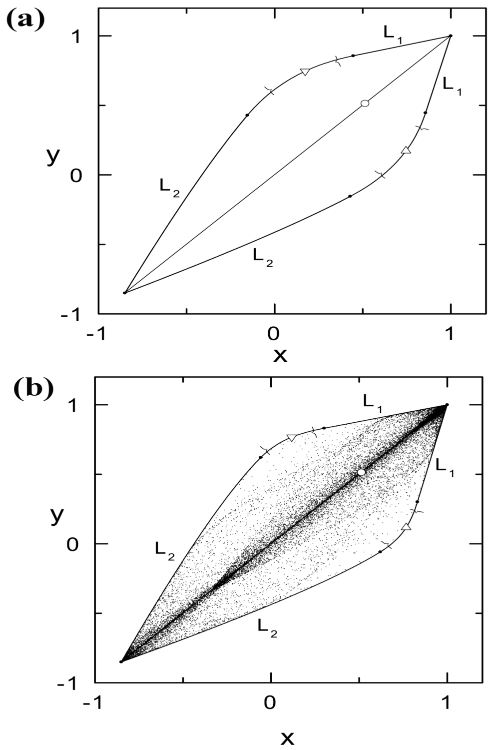

Fig. 3. Symmetry-conserving blow-out bifurcation for $a = 1.85$ in the dissipative-coupling case. (a) A symmetric mixed absorbing area for $c = -1.47$ is bounded by a union of segments of the critical curves L_1 and L_2 and segments of unstable manifolds of a conjugate pair of asymmetric saddle fixed points ($\triangle$ and $\bigtriangledown$) born from the synchronous fixed point ($\bigcirc$) via a pitchfork bifurcation. (b) A symmetric CA for $c = -1.49$, born via a symmetry-conserving blow-out bifurcation.

through a parameter mismatching, we can obtain the minimal invariant absorbing area bounded by segments of the critical curves L_k $(k = 1, 2, 3, 4)$ for $\epsilon = 0$, which is shown in Fig. 1(b). Note that the absorbing area, including the CA with two bands on the synchronization diagonal, is a symmetric one. As c passes $c_{b,r}$, the synchronous CA becomes transversely unstable and transforms to a chaotic saddle with $\sigma_\perp > 0$. Hence, a typical trajectory starting near the synchronization diagonal is spread over the entire symmetric absorbing area. Figure 1(c) shows a symmetric asynchronous CA for $c = -1.024$. Consequently, for this case, the symmetry is conserved when the blow-out bifurcation occurs (*i.e.*, a symmetry-conserving blow-out bifurcation takes place).

Figure 2(a) shows a synchronous CA with four bands on the diagonal for $a = 1.427$ and $c = -1.031$. As above, through a small parameter mismatching, we get information on the shape of the minimal invariant absorbing area, including the synchronous CA, and then construct the minimal invariant absorbing area bounded by segments of the critical curves L_k $(k = 1, ..., 32)$. For clear presentation, only a part of the whole absorbing area, including the upper two chaotic bands, is shown in Fig. 2(b). Note that this absorbing area is an asymmetric one, in contrast to the above symmetric case. (In fact, there exists a conjugate pair of asymmetric absorbing areas due to the exchange symmetry.) Obviously, when crossing a blow-out bifurcation point c_b $(= -1.029)$, a symmetry-breaking blow-out bifurcation occurs, which results in the birth of a conjugate pair of asymmetric CAs. An asymmetric CA for $c = -1.027$ is shown in Fig. 2(c).

In addition to the linear-coupling case, the type of blow-out bifurcations is also studied in the dissipatively-coupled case with $g(x, y) = c(y^2 - x^2)$, and only symmetry-conserving blow-out bifurcations are observed for all cases studied. As an example, consider the case of $a = 1.85$, where a chaotic state with a single band exists on the diagonal. For $c = -1.47$ (just before the blow-out bifurcation), a union of segments of the critical curves L_1 and L_2 and segments of unstable manifolds of a conjugate pair of asymmetric saddle fixed points (denoted by the up and the down triangles) born from the synchronous fixed point (denoted by the circle) via a pitchfork bifurcation is used to define a symmetric mixed absorbing area, as shown in Fig. 3(a). Consequently, when c passes a threshold point c_t $(= -1.485)$, a symmetry-conserving blow-out bifurcation occurs, leading to the birth of a symmetric CA, which is shown in Fig. 3(b).

To summarize, we have investigated the conservation and the breakdown of the symmetry when a blow-out bifurcation occurs. It has been found that the shape of the minimal invarsiant absorbing area determines the type of blow-out bifurcation. For the case of a symmetric absorbing area, a symmetry-conserving blow-out bifurcation occurs while in the case of an asymmetric absorbing area, a symmetry-breaking blow-out bifurcation takes place. Just after the blow-out bifurcations, both the symmetric and the asymmetric asynchronous CAs exhibit typical on-off intermittent behaviors [8].

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