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Intermittent Strange Nonchaotic Attractors in Quasiperiodically Forced Systems

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Intermittent strange nonchaotic attractors (SNAs) appear typically in quasiperiodically forced systems. As a basic model, we consider the quasiperiodically forced Hénon map and investigate the mechanism for the intermittent transition to SNAs. Using rational approximations to the quasiperiodic forcing, it is shown that dynamical transition to an intermittent SNA occurs via a phase-dependent saddle-node bifurcation when a smooth torus collides with a new kind of invariant "ring-shaped" unstable set.

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Strange nonchaotic attractors (SNAs) typically appear in quasiperiodically forced dynamical systems [1]. These attractors were first described by Grebogi *et al.* [2] and have been extensively investigated both theoretically [3–15] and experimentally [16]. SNAs exhibit some properties of regular, as well as chaotic attractors. Like regular attractors, their dynamics is nonchaotic in the sense that they do not have a positive Lyapunov exponent; like usual chaotic attractors, they have a geometrically strange (fractal) structure. In recent years, dynamical transitions to SNAs have become a topic of considerable current interest.

Here, we are interested in the dynamical transition to SNAs accompanied by intermittent behavior [11]. As a parameter passes a threshold value, a smooth torus abruptly transforms to an intermittent SNA. This transition to an intermittent SNA is quite general, and has been observed in a number of quasiperiodically forced systems (*e.g.*, see [12, 13]). Recently, in Ref. 15, we have investigated the bifurcation mechanism for the intermittent transition to SNAs in a simple model of the quasiperiodically forced (noninvertible) logistic map.

As a representative model for the Poincaré maps of continuous-time systems forced at two incommensurate frequencies, we consider the quasiperiodically forced (invertible) Hénon map [6]:

$$M : \begin{cases} x_{n+1} = a - x_n^2 + y_n + \varepsilon \cos 2\pi\theta_n, \\ y_{n+1} = b x_n, \\ \theta_{n+1} = \theta_n + \omega \pmod 1, \end{cases} \tag{1}$$

where a is the nonlinearity parameter of the unforced Hénon map, and ω and ε represent the frequency and amplitude of the quasiperiodic forcing, respectively. This quasiperiodically forced Hénon map M is invertible, because it has a nonzero constant Jacobian determinant $-b$ whose magnitude is less than unity (*i.e.*, $b \neq 0$ and $-1 < b < 1$). Here, we fix the value of the dissipation parameter b at $b = 0.05$.

In this paper, the frequency ω is set to be the reciprocal of the golden mean, $\omega = (\sqrt{5} - 1)/2$. Then, using the RAs to this quasiperiodic forcing, we investigate the intermittent transition to SNAs. For the inverse golden mean, its rational approximants are given by the ratios of the Fibonacci numbers, $\omega_k = F_{k-1}/F_k$, where the sequence of $\{F_k\}$ satisfies $F_{k+1} = F_k + F_{k-1}$ with $F_0 = 0$ and $F_1 = 1$. Instead of the quasiperiodically forced system, we study an infinite sequence of periodically forced systems with rational driving frequencies ω_k. We suppose that the properties of the original system M may be obtained by taking the quasiperiodic limit $k \to \infty$.

Fig. 1(a) shows a phase diagram in the $a - \varepsilon$ plane. Each phase is characterized by the (nontrivial) Lyapunov exponents, σ_1 and σ_2 ($\leq \sigma_1$), associated with dynamics of the variables x and y (besides the zero exponent, connected to the phase variable θ of the quasiperiodic forcing) as well as the phase sensitivity exponent δ. The exponent δ measures the sensitivity with respect to the phase of the quasiperiodic forcing and characterizes the strangeness of an attractor [5]. A smooth torus has negative Lyapunov exponents ($\sigma_{1,2} < 0$) and has no phase sensitivity (*i.e.*, $\delta = 0$). The region where it exists is denoted by T and shown in light gray. Upon crossing the solid line, the smooth torus becomes unstable and bifurcates to a smooth doubled torus in the region denoted by $2T$. On the other hand, a chaotic attractor has a positive Lyapunov exponent $\sigma_1 > 0$, and its region is shown in black. Between these regular and chaotic regions,

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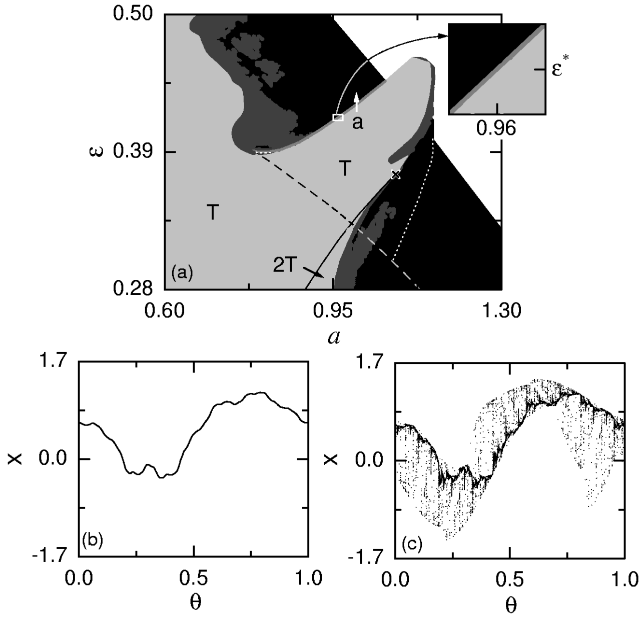

Fig. 1. (a) Phase Diagram in the $a - \varepsilon$ plane for the case $b = 0.05$ and $\omega = (\sqrt{5} - 1)/2$. Regular, chaotic, SNA and divergence regimes are shown in light gray, black, gray (or dark gray), and white, respectively. To show the region of existence (gray) of the intermittent SNA occurring between T (light gray) and the chaotic attractor region (black), a small box near $(a, \varepsilon) = [0.96, \varepsilon^*(= 0.416\,857\,986)]$ is magnified. Here, the torus and the doubled torus are denoted by T and $2T$, and the solid line represents a torus-doubling bifurcation curve whose terminal point is marked with the cross. Through interaction with the ring-shaped unstable set born when passing the dashed line, dynamical transition to an intermittent SNA (route a) occurs. A projection of a(n) smooth torus (intermittent SNA) onto the $\theta - x$ plane for $a = 0.96$ and $\varepsilon = 0.415$ ($\varepsilon = 0.416\,86$) is shown in (b) [(c)]. For other details, see the text.

SNAs that have negative Lyapunov exponents ($\sigma_{1,2} < 0$) and positive phase sensitivity exponents ($\delta > 0$) exist in the regions shown in gray and dark gray. Due to their high phase sensitivity, SNAs are observed to have a strange fractal structure. In the thin gray region [*e.g.*, see a magnified part in Fig. 1(a)], intermittent SNAs exist, while in the dark-gray region nonintermittent SNAs, born through the mechanism of gradual fractalization [10] or torus collision [4], exist.

A particularly interesting feature of the phase diagram is the existence of the "tongue" of quasiperiodic motion that penetrates into the chaotic region and separates it into upper and lower parts. This tongue lies near the terminal point (denoted by the cross) of the torus-doubling bifurcation curve. When crossing the upper boundary of the tongue, a smooth torus transforms to an intermittent SNA that exists in the thin gray region. Here, we investigate this intermittent route to SNAs [see the route a in Fig. 1(a)]. As an example, consider the case of $a = 0.96$. Figure 1(b) shows a projection of a smooth torus with $\sigma_1 = -0.077$ onto the $\theta - x$ plane for $\varepsilon = 0.415$. We note that the projection is a smooth invariant curve. A curve can be regarded as a cross section (Poincaré map) for two-frequency quasiperiodic motion on a smooth torus in continuous-time dynamical systems. Hence, we call this curve a torus. However, as ε passes a threshold value ε^* ($= 0.416\,857\,986$), dynamical transition to an intermittent SNA, occupying a finite volume of the phase space, occurs. As shown in Fig. 1(c) for $\varepsilon = 0.416\,86$, a typical trajectory on the newly-born intermittent SNA with $\sigma_1 = -0.006$ and $\delta = 4.32$ spends most of its time near the former torus, with sporadic large bursts away from it.

By using RAs, we find a new type of invariant ring-shaped unstable set that causes the intermittent transition through a collision with the smooth torus. When passing the dashed curve in Fig. 1(a), such a ring-shaped unstable set appears through a phase-dependent saddle-node bifurcation which has no counterpart in the unforced case. (The dashed line is obtained for a sufficiently large level $k = 10$ of the RAs.) For each RA of level k, a periodically forced Hénon map with rational driving frequency ω_k has a periodic or a chaotic attractor that depends on the initial phase θ_0 of the external forcing. Then, the union of all attractors for different θ_0 gives the kth approximation to the attractor in the quasiperiodically forced system.

As an example, consider the RA of level $k = 7$. The RA to the smooth torus (denoted by a black line), consisting of stable orbits with period F_7 ($= 13$), is shown in Fig. 2(a) for $a = 0.85$ and $\varepsilon = 0.3707$. We also note that a ring-shaped unstable set, born via a phase-dependent saddle-node bifurcation and composed of 13 small rings, lies near the RA to the smooth torus. At first, each ring consists of the stable (shown in black) and unstable (shown in gray) orbits with forcing period F_7 [see the inset in Fig. 2(a)]. However, as the parameters increase such rings evolve, as shown in Fig. 2(b) for $a = 0.86$ and $\varepsilon = 0.375$. For fixed values of a and ε, the phase θ may be regarded as a "bifurcation parameter." As θ changes, a chaotic attractor appears through an infinite sequence of period doubling bifurcations of stable periodic orbits in each ring, and then it disappears through collision with the unstable F_7-periodic orbit [see the inset in Fig. 2(b)]. Thus, the attracting part (shown in black) of each ring consists of the union of the originally stable F_7-periodic attractor and the higher $2^n F_7$-periodic ($n = 1, 2, \ldots$) and chaotic attractors born through the period-doubling cascade. On the other hand, the unstable part (shown in gray) of each ring is composed of the union of the originally unstable F_7-periodic orbit [*e.g.*, the upper gray line in the inset in Fig. 2(b)] born via a saddle-node bifurcation and the destabilized F_7-periodic orbit [*e.g.*, the lower gray line in the inset in Fig. 2(b)] born through a period doubling bifurcation. (As will be seen below, only the originally unstable F_7-periodic orbit may interact with the stable F_7-periodic orbit in the RA to the smooth torus through a saddle-node bifurcation.) As the pa-

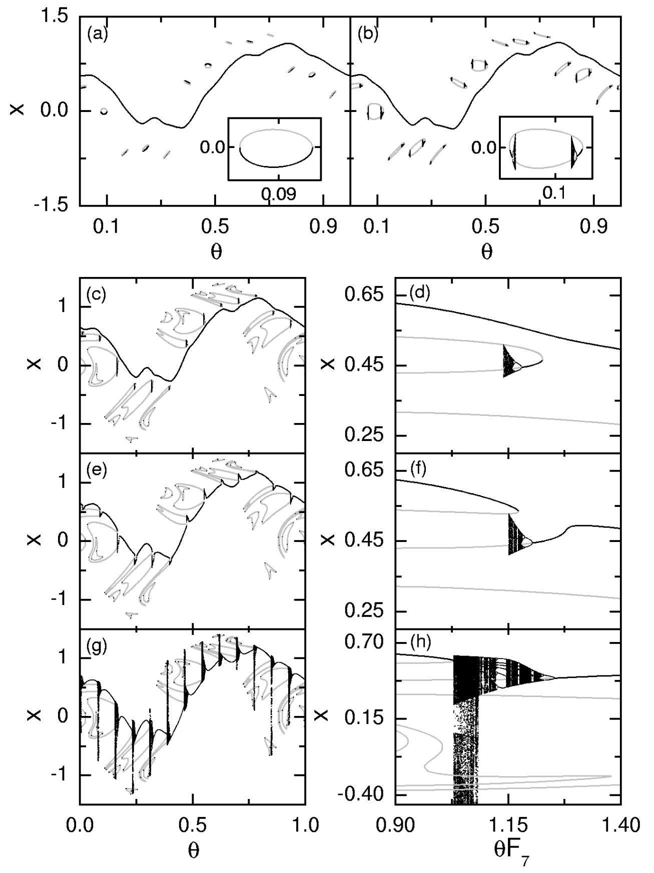

Fig. 2. Dynamical mechanism for the intermittent transition to SNAs. Smooth torus (denoted by a black curve) and ring-shaped unstable set (composed of 13 rings) in the RA of level 7 for (a) $a = 0.85$ and $\varepsilon = 0.3707$ and (b) $a = 0.86$ and $\varepsilon = 0.375$. Each ring is composed of the attracting part (shown in black) and the unstable part (shown in gray and consisting of unstable F_7-periodic orbits). In (c), the ring-shaped unstable set (composed of 26 rings) lies close to the smooth torus (denoted by a black curve) for $a = 0.96$ and $\varepsilon = 0.4$. A magnified view near $(\theta F_7, x) = (1.15, 0.45)$ is given in (d). For this case, the intermittent transition from a smooth torus to an SNA occurs through the following two procedures. First, the RA to the attractor becomes nonsmooth via a phase-dependent saddle-node bifurcation, as shown in (e) and (f) for $a = 0.96$ and $\varepsilon = 0.4015$. Second, the chaotic component in the RA to the attractor becomes suddenly widened via an interior crisis, as shown in (g) and (h) for $a = 0.96$ and $\varepsilon = 0.4045$. This RA to the intermittent SNA is composed of the union of the periodic component and the intermittent chaotic component, where the latter occupies the 13 "gaps" in θ. For more details, see the text.

rameters are further increased, both the size and shape of the rings change, and for sufficiently large parameters, each ring consists of a large unstable part (shown in gray) and a small attracting part (shown in black), as shown in Fig. 2(c) for $a = 0.96$ and $\varepsilon = 0.4$. Furthermore, new rings may appear inside or outside the "old" rings via another (phase-dependent) saddle-node bifurcation [*e.g.*, see 13 new small rings in Fig. 2(c).] As the level k increases, the ring-shaped unstable set consists of a larger number of rings with a smaller attracting part (*i.e.*, as the level k is increased, the unstable part of each ring becomes more and more dominant) [15]. Hence, it is conjectured that, in the quasiperiodic limit, these ring-shaped unstable sets might form a complicated invariant unstable set composed of only unstable orbits.

In terms of the RA of level 7, we explain the mechanism for the intermittent transition occurring in Figs. 1(b) and 1(c) for $a = 0.96$. As we approach the border of the intermittent transition in the phase diagram, the ring-shaped unstable set comes closer to the smooth torus (denoted by a black curve), as shown in Figs. 2(c) and 2(d). As ε passes a threshold value $\varepsilon_7^{(1)}$ ($= 0.401\,035\,615$), a phase-dependent saddle-node bifurcation occurs between the smooth torus and the unstable part (shown in gray) of the ring-shaped unstable set. Then, the new attractor of the system contains the attracting part (shown in black) of the ring-shaped unstable set and becomes nonsmooth [*e.g.*, see Figs. 2(e) and 2(f) for $\varepsilon = 0.4015$]. As ε is further increased, the chaotic component in the RA to the attractor increases, and eventually for $\varepsilon_7^{(2)} = 0.403\,399\,486$, it becomes suddenly widened via an interior crisis when it collides with the nearest ring [*e.g.*, see Fig. 2(g) for $\varepsilon = 0.4045$]. Then, "gaps," where no attractors with period F_7 exist and all the rings are contained, appear. A magnified gap is shown in Fig. 2(h) for $\varepsilon = 0.4045$. Note that this gap is filled by intermittent chaotic attractors together with orbits with period higher than F_7 embedded in very small windows. As shown in Fig. 2(g), the RA to the whole attractor consists of the union of the periodic component and the intermittent chaotic component, where the latter occupies the 13 gaps in θ. For this case, the periodic component dominates, and hence the average 1st Lyapunov exponent ($\langle\sigma_1\rangle = -0.168$) becomes negative, where $\langle\cdots\rangle$ denotes the average over the whole θ. We note that Fig. 2(g) resembles Fig. 1(c), although the level $k = 7$ is low. Thus, in the RAs the intermittent transition to an SNA consists of two stages: the phase-dependent saddle-node bifurcation and the interior crisis. On increasing the level to $k = 18$, we find the threshold values $\varepsilon_k^{(1)}$ and $\varepsilon_k^{(2)}$ at which the phase-dependent saddle-node bifurcation and the interior crisis occur, respectively. As the level k increases, the difference $\Delta\varepsilon_k$ [$\equiv \varepsilon_k^{(2)} - \varepsilon_k^{(1)}$] tends to zero, and both sequences of $\{\varepsilon_k^{(1)}\}$ and $\{\varepsilon_k^{(2)}\}$ converge to the same quasiperiodic limit ε^* ($= 0.416\,857\,986$) in an algebraic manner; $\Delta\varepsilon_k^{(i)}$ [$\equiv \varepsilon_k^{(i)} - \varepsilon^*$] $\sim F_k^{-\alpha}$ ($i = 1, 2$), where $\alpha \simeq 2.0$. In the quasiperiodic limit $k \to \infty$, the RA to the attractor has a dense set of gaps which are filled by intermittent chaotic attractors. Thus, an intermittent SNA, containing the ring-shaped unstable set, appears, as shown in Fig. 1(c).

In summary, by using the RAs we have investigated the intermittent transition to SNAs in the quasiperiodically forced (invertible) Hénon map. It has been found that, when a smooth torus makes a collison with a ring-shaped unstable set, dynamical transition to an intermittent SNA occurs via a phase-dependent saddle-node bifurcation. These kinds of intermittent transition have also been observed in other quasiperiodically forced invertible systems, such as the quasiperiodically forced ring map and Toda oscillator [17]. We also note that the bifurcation mechanism for the appearance of intermittent SNAs is the same as that in the quasiperiodically forced (noninvertible) logistic map [15]. Hence, the intermittent route to SNAs seems to be a generic transition, occurring through the same mechanism in typical quasiperiodically forced systems of different nature.

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REFERENCES

[1] A. Prasad, S. S. Negi and R. Ramaswamy, Int. J. Bif. Chaos **11**, 291 (2001).

[2] C. Grebogi, E. Ott, S. Pelikan and J. A. Yorke, Physica D **13**, 261 (1984).

[3] F. J. Romeiras and E. Ott, Phys. Rev. A **35**, 4404 (1987); M. Ding, C. Grebogi and E. Ott, Phys. Rev. A **39**, 2593 (1989).

[4] J. F. Heagy and S. M. Hammel, Physica D **70**, 140 (1994).

[5] A. S. Pikovsky and U. Feudel, Chaos **5**, 253 (1995). See Eqs. (11)-(14) for the definition of the phase sensitivity exponent δ.

[6] O. Sosnovtseva, U. Feudel, J. Kurths and A. Pikovsky, Phys. Lett. A **218**, 255 (1996).

[7] U. Feudel, J. Kurths and A. S. Pikovsky, Physica D **88**, 176 (1995); U. Feudel, C. Grebogi and E. Ott, Phys. Rep. **290**, 11 (1997).

[8] S. P. Kuznetsov, A. S. Pikovsky and U. Feudel, Phys. Rev. E **51**, R1629 (1995); S. Kuznetsov, U. Feudel and A. Pikovsky, Phys. Rev. E **57**, 1585 (1998); S. Kuznetsov, E. Neumann, A. Pikovsky and I. Sataev, Phys. Rev. E **62**, 1995 (2000); S. P. Kuznetsov, Phys. Rev. E **65**, 066209 (2002).

[9] P. R. Chastell, P. A. Glendinning and J. Stark, Phys. Lett. A **200**, 17 (1995); H. Osinga, J. Wiersig, P. Glendinning and U. Feudel, Int. J. Bif. Chaos **11**, 3085 (2001).

[10] T. Nishikawa and K. Kaneko, Phys. Rev. E **54**, 6114 (1996).

[11] A. Prasad, V. Mehra and R. Ramaswamy, Phys. Rev. Lett. **79**, 4127 (1997); Phys. Rev. E **57**, 1576 (1998); S.-Y. Kim, J. Korean Phys. Soc. **34**, 75 (1999).

[12] A. Witt, U. Feudel and A. Pikovsky, Physica D **109**, 180 (1997).

[13] A. Venkatesan, M. Lakshmanan, A. Prasad and R. Ramaswamy, Phys. Rev. E **61**, 3641 (2000); A. Venkatesan and M. Lakshmanan, Phys. Rev. E **63**, 026219 (2001).

[14] B.R. Hunt and E. Ott, Phys. Rev. Lett. **87**, 254101 (2001); J.-W. Kim, S.-Y. Kim, B. Hunt and E. Ott, Phys. Rev. E **67**, 036211 (2003).

[15] S.-Y. Kim, W. Lim and E. Ott, Phys. Rev. E **67**, 056203 (2003).

[16] W. L. Ditto, M. L. Spano, H. T. Savage, S. N. Rauseo, J. Heagy and E. Ott, Phys. Rev. Lett. **65**, 533 (1990); T. Zhou, F. Moss and A. Bulsara, Phys. Rev. A **45**, 5394 (1992); W. X. Ding, H. Deutsch, A. Dinklage and C. Wilke, Phys. Rev. E **55**, 3769 (1997); T. Yang and K. Bilimgut, Phys. Lett. A **236**, 494 (1997); B. P. Bezruchko, S. P. Kuznetsov and Y. P. Seleznev, Phys. Rev. E **62**, 7828 (2000); Y.-H. Yu, D.-C. Kim, J.-Y. Ryu, S.-R. Hong, K. Kwak, T.-K. Lim and W. Jung, J. Korean Phys. Soc. **34**, 130 (1999).

[17] W. Lim and S.-Y. Kim (unpublished).