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Mechanism for the Band-Merging Route to Strange Nonchaotic Attractors in Quasiperiodically Forced Systems

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As a representative model for quasiperiodically forced period-doubling systems, we consider the quasiperiodically forced Hénon map and investigate the dynamical mechanism for the band-merging route to intermittent strange nonchaotic attractors (SNAs). Using the rational approximation to quasiperiodic forcing, we show that a band-merging transition from a two-band smooth torus to a single-band intermittent SNA occurs when the smooth torus collides with a ring-shaped unstable set which has no counterpart in the unforced case. The mechanism for the band-merging transition to intermittent SNAs is also confirmed in the quasiperiodically forced Toda oscillator. In addition to inducing the transition to SNAs, such a band-merging mechanism is a direct cause for the truncation of the torus-doubling sequence.

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I. INTRODUCTION

Strange nonchaotic attractors (SNAs) typically appear in quasiperiodically forced dynamical systems [1]. They exhibit some properties of regular, as well as chaotic, attractors. Like regular attractors, their dynamics is nonchaotic in the sense that they do not have a positive Lyapunov exponent; like usual chaotic attractors, they have a geometrically strange (fractal) structure. These SNAs were first described by Grebogi *et al.* [2] and have been extensively investigated both theoretically [3–17] and experimentally [18]. Moreover, they are related to the Anderson localization in the Schrödinger equation with a quasiperiodic potential [19], and may have a practical application in secure communication [20]. Hence, dynamical routes to SNAs have become a topic of considerable current interest [4, 7–13, 15–17]. However, the mechanisms for the birth of SNAs are much less clear than those for the appearance of chaotic attractors in periodically forced systems.

Here, we are interested in the band-merging route to intermittent SNAs. As a parameter passes a threshold value, a two-band smooth torus, which is born from its single-band parent torus via a torus-doubling bifurcation, abruptly transforms into a single-band intermittent SNA, and then the torus-doubling sequence is truncated. This band-merging transition to an intermittent SNA is quite general and has been observed in a number of quasiperiodically forced period-doubling systems (*e.g.*, see Refs. [10–12]). However, the unstable orbit inducing such a band-merging transition was not located; hence, the dynamical origin for the band-merging transition remains unclear.

This paper is organized as follows: In Sec. II, we investigate the mechanism for the band-merging route to an intermittent SNA in the quasiperiodically forced Hénon map which is a representative model for quasiperiodically forced period-doubling maps. We use the rational approximation to quasiperiodic forcing and find that a transition from a two-band smooth torus to a single-band intermittent SNA occurs through a collision with an invariant ring-shaped unstable set which has no counterpart in the unforced case. The dynamical mechanism for the band-merging transition is also confirmed in the quasiperiodically forced Toda oscillator, which is another representative model for quasiperiodically forced period-doubling flows. We note that this kind of band-merging route to intermittent SNAs is in contrast to the Heagy-Hammel route to SNAs via torus collision [4]. For the latter case, a two-band smooth torus is transformed into a single-band SNA, exhibiting no intermittency, when it collides with its unstable parent torus. Both mechanisms for the birth of SNAs are direct causes for the interruption of torus-doubling cascades [21]. Finally, a summary is given in Sec. III.

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II. BAND-MERGING TRANSITION TO INTERMITTENT STRANGE NONCHAOTIC ATTRACTORS

We consider the quasiperiodically forced Hénon map, which is a representative model for quasiperiodically forced period-doubling systems:

$$M : \begin{cases} x_{n+1} = a - x_n^2 + y_n + \varepsilon \cos 2\pi\theta_n, \\ y_{n+1} = bx_n, \\ \theta_{n+1} = \theta_n + \omega \pmod 1), \end{cases} \qquad (1)$$

where a is the nonlinearity parameter of the unforced Hénon map, and ω and ε represent the frequency and the amplitude of the quasiperiodic forcing, respectively. This quasiperiodically forced Hénon map M is invertible because it has a nonzero constant Jacobian determinant $-b$ whose magnitude is less than unity (*i.e.*, $b \neq 0$ and $-1 < b < 1$). Here, we fix the value of the dissipation parameter b at $b = 0.05$ and set the frequency ω to be the reciprocal of the golden mean, $\omega = (\sqrt{5} - 1)/2$. Then, using the rational approximation to this quasiperiodic forcing, we investigate the dynamical mechanism for the band-merging route to intermittent SNAs. For the inverse golden mean, its rational approximants are given by the ratios of the Fibonacci numbers, $\omega_k = F_{k-1}/F_k$, where the sequence of $\{F_k\}$ satisfies $F_{k+1} = F_k + F_{k-1}$ with $F_0 = 0$ and $F_1 = 1$. Instead of a quasiperiodically forced system, we study an infinite sequence of periodically forced systems with rational driving frequencies ω_k and suppose that the properties of the original system M may be obtained by taking the quasiperiodic limit $k \to \infty$.

Fig. 1(a) shows a phase diagram in the $a - \varepsilon$ plane. Each phase is characterized by the (nontrivial) Lyapunov exponents σ_1 and σ_2 ($\leq \sigma_1$) associated with the dynamics of the variables x and y (besides the zero exponent, connected to the phase variable θ of the quasiperiodic forcing) as well as the phase sensitivity exponent δ. The exponent δ measures the sensitivity with respect to the phase of quasiperiodic forcing and characterizes the strangeness of an attractor [5]. A two-band smooth torus, which is born via a first-order torus-doubling bifurcation of its parent torus with a single band, exists in the region represented by $2T$ and is shown in light gray. It has negative Lyapunov exponents ($\sigma_{1,2} < 0$) and no phase sensitivity ($\delta = 0$). When crossing the solid line (corresponding to a second-order torus-doubling bifurcation line), the two-band smooth torus becomes unstable and bifurcates to a four-band smooth torus, which exists in the region denoted by $4T$. On the other hand, a chaotic attractor with a positive Lyapunov exponent ($\sigma_1 > 0$) exists in the region shown in black. Between these regular and chaotic regions, SNAs that have negative Lyapunov exponents ($\sigma_{1,2} < 0$) and high phase sensitivity ($\delta > 0$) exist in the region shown in gray. Because of their high phase sensitivity, these SNAs have fractal structure [5]. A very interesting feature of the phase diagram is the existence of a second-order "tongue" that

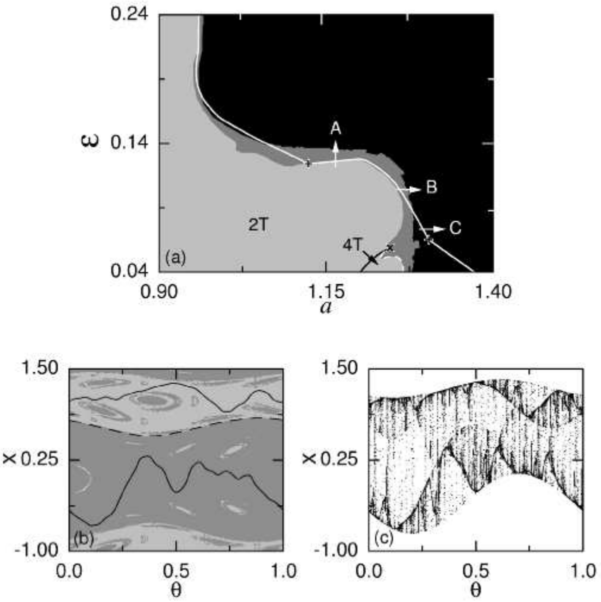

Fig. 1. (a) Phase Diagram of the quasiperiodically forced Hénon map M in the $a - \varepsilon$ plane for the case of $b = 0.05$ and $\omega = (\sqrt{5}-1)/2$. Regular, chaotic, and SNA regions are shown in light gray, black, and gray, respectively. For the case of a regular attractor, tori with two and four bands exist in the regions denoted by $2T$ and $4T$, respectively. When crossing the white solid curve, a two-band attractor is transformed into a single-band attractor; a transition from a four-band attractor to a two-band attractor occurs when passing the white dashed curve. Particularly, a band-merging transition to an intermittent SNA takes place along route A. In (b) and (c), projections of the attractors and of the smooth unstable torus (represented by a dashed curve) onto the $\theta - x$ plane and the 2D slices with $y = 0.005$ of the basins of the attractors are given for $a = 1.17$. There exists a pair of conjugate tori in M^2, which are denoted by black curves in (b) for $\varepsilon = 0.12$. The basins of the upper and the lower tori are shown in light gray and gray, respectively. Through a collision with a hole boundary, the conjugate tori merge into a single-band intermittent SNA in M, as shown in (c) for $\varepsilon = 0.127$.

penetrates into the chaotic region. This tongue lies near the terminal point (denoted by the cross) of the second-order torus-doubling bifurcation curve, as in the case of the main (first-order) tongue that exists near the terminal point of the first-order torus-doubling bifurcation line (*e.g.*, see Fig. 1(a) in Ref. [15]).

When passing the solid white line in Fig. 1(a), a two-band attractor (smooth torus, SNA, or chaotic attractor) transforms into a single-band attractor. This band-merging curve starts from the band-merging point for the unforced case ($\varepsilon = 0$) where a two-band chaotic attractor in the Hénon map transforms smoothly to a single-band chaotic attractor through a collision with its parent unstable fixed point. As ε is increased from zero, a natural generalization of the band merging for the unforced case occurs for a two-band chaotic attractor when it collides with the smooth unstable torus that is developed from the unstable fixed point of the (unforced) Hénon map. Hereafter, we will call this generalized transition from the unforced case as the "standard" band-merging transition. Thus, the standard band-merging transition curve continues smoothly in the (a, ε) plane. However, it loses its differentiability at the two vertices denoted by the pluses (+). Then, a nonstandard band-merging transition occurs along the routes A, B, and C crossing the segment bounded by the two vertices. [Above the upper vertex, standard band mergings occur again for a nonchaotic attractor (smooth torus and SNA) as well as a chaotic attractor.] As in the case of interior crisis [22], the nonstandard band merging of a two-band attractor is a kind of "hard" transition, because the attractor size is suddenly increased through an abrupt merging of the two separate bands without direct touching. This is in contrast to the "soft" transition occurring for the case of the standard band merging where the two bands join smoothly without any abrupt increase of the attractor size. For the case of nonstandard band mergings, due to a basin boundary metamorphosis [23], the smooth unstable torus becomes inaccessible from the interior of the basin of the attractor, and hence it cannot induce any band merging. For this case, through a collision with a ring-shaped unstable set which has no counterpart for the unforced case, a nonstandard band-merging transition occurs for a nonchaotic attractor [smooth torus (route A) or SNA (route B)] as well as a chaotic attractor (route C), as will be shown below. Particularly, a two-band smooth torus transforms into a single-band intermittent SNA along the route A, which corresponds to a mechanism for the appearance of SNAs. Here, we are interested in this type of band-merging route to intermittent SNAs, and investigate its dynamical mechanism by using the rational approximation to the quasiperiodic forcing. (Although a similar case where a three-band smooth torus abruptly transforms to a single-band intermittent SNA was observed in the study on the effect of quasiperiodic forcing on the interior crisis occurring for unforced case ($\varepsilon = 0$) [11], the unstable orbit inducing such a transition was not explicitly located.)

As an example, we consider the case of $a = 1.17$ and study the band-merging transition from a two-band torus to a single-band intermittent SNA by varying ε along route A. It is convenient to investigate such a band-merging transition in M^2 (*i.e.*, the second iterate of the original map M). A two-band smooth torus in M is transformed into a pair of conjugate tori in M^2. Fig. 1(b) shows a pair of upper and lower tori (denoted by black curves) for $\varepsilon = 0.12$, whose basins are shown in light gray and gray, respectively. For this case, the basin of each smooth torus contains "holes" of the other basin of the counterpart. Hence, the smooth unstable torus (denoted by the dashed line) on a basin boundary is not accessible from the interiors of the basins of the conjugate attracting tori, so it cannot induce any band-merging transition. For this case of a basin boundary metamorphosis, conjugate tori and holes become closer as the parameter ε increases. Eventually, for $\varepsilon = \varepsilon^*$ $(= 0.126\,662\,718)$ an attractor-merging crisis occurs for the conjugate tori via a collision with a hole boundary; then, a single-band intermittent SNA appears in the original map M. As shown in Fig. 1(c) for $\varepsilon = 0.127$, a typical trajectory on the newly-born intermittent SNA with $\sigma_1 \simeq -0.029$ and $\delta \simeq 4.60$ spends most of its time near the former two-band torus with sporadic large bursts away from it.

Such a band-merging transition to an intermittent SNA takes place through a collision with a ring-shaped unstable set on a hole boundary, as will be shown below. Using the rational approximation, the ring-shaped unstable set, which has no counterpart in the unforced case, was first discovered in a study of the intermittent route to SNAs [15]. As the system parameters vary, both the sizes and the shapes of rings constituting the unstable set are changed. Furthermore, as the level of the rational approximation increases, the ring-shaped unstable set consists of a large number of rings; hence, it becomes a complicated unstable set. (For details on the structure and the evolution of a ring-shaped unstable set, refer to Fig. 2 of Ref. [15].)

In terms of the rational approximation of level 8, we now explain the mechanism for the band-merging route to intermittent SNAs occurring in Figs. 1(b)-1(c) for $a = 1.17$. Fig. 2(a) shows conjugate tori (denoted by black curves), conjugate ring-shaped unstable sets (represented by dark gray curves), and holes (shown in gray and light gray inside the basins of the upper and the lower tori, respectively) in M^2 for $\varepsilon = 0.125$. The rational approximations to the conjugate smooth tori and the ring-shaped unstable sets are composed of stable and unstable orbits with period F_8 $(= 21)$ in M^2, respectively. For this case, some part of each ring-shaped unstable set (denoted by dark gray curves) lies on a hole boundary (*e.g.*, see a magnified view in Fig. 2(b), where holes in the light gray basin are represented by gray dots). With an increase in ε, the conjugate tori and the ring-shaped unstable sets become closer, and eventually, for $\varepsilon = \varepsilon_8^*$ $(= 0.125\,669\,395)$, a pair of phase-dependent saddle-node bifurcations occurs for the conjugate stable and unstable F_8-periodic orbits through collision between the conjugate tori and the ring-shaped unstable sets. Then, F_8 $(= 21)$ "gaps," where no orbits with period F_8 exist, are formed over the whole range of θ, as shown in Fig. 2(c) for $\varepsilon = 0.1257$ [*e.g.*, see the magnified gap in Fig. 2(d)]. In these gaps, single-band intermittent chaotic attractors (denoted by black dots) appear (*i.e.*, saddle-node bifurcations induce attractor-merging crises in the gaps). Thus, the rational approximation to the whole attractor in the original map M becomes composed of the union of the two-band periodic component and the single-band intermittent chaotic component. Since the periodic component is dominant, its average Lyapunov exponent ($\langle \sigma_1 \rangle \simeq -0.098$) is negative, where $\langle \cdots \rangle$ denotes the average over all θ. Hence, the partially-merged 8th rational approximation to the at-

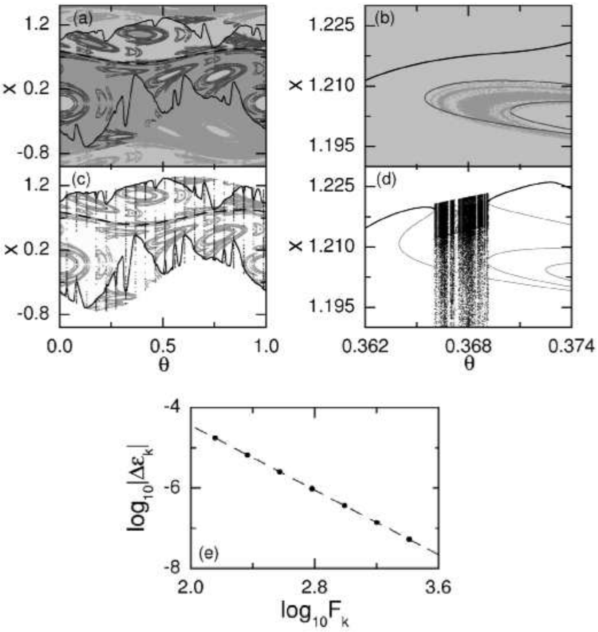

Fig. 2. Dynamical mechanism for the transition from a two-band smooth torus to a single-band intermittent SNA for $a = 1.17$ in the quasiperiodically forced Hénon map M. In (a)-(d), projections of the attractors, the ring-shaped unstable sets, and the smooth unstable torus (denoted by a dashed curve) onto the $\theta - x$ plane and the 2D slices with $y = 0.005$ of the basins of the attractors are given in the rational approximation of level 8. (a) and (b) Eighth rational approximation to the conjugate smooth tori and ring-shaped unstable sets for $\varepsilon = 0.125$ in M^2. The basins of the upper and the lower tori, denoted by black curves, are shown in light gray and gray, respectively. The conjugate ring-shaped unstable sets, represented by dark gray curves, lie close to the smooth tori (*e.g.*, see the magnified view in (b), where the holes in the light gray basin are denoted by gray dots). (c) and (d) Eighth rational approximation to the single-band intermittent SNA for $\varepsilon = 0.1257$ in M. The attractors and the ring-shaped unstable sets are shown in black and gray, respectively. The rational approximation to the SNA is composed of the union of the two-band periodic component and the single-band intermittent chaotic component, where the latter occupies the F_8 (= 21) gaps in θ. For a clear view, a magnified gap is given in (d). (e) Plot of $\log_{10}|\Delta\varepsilon_k^*|$ vs. $\log_{10} F_k$ for $k = 12, \ldots, 18$ [$\Delta\varepsilon_k^* = \varepsilon_k^* - \varepsilon^*$]. Here, ε_k^* (denoted by solid circles) represents the threshold value for the saddle-node bifurcation (inducing the attractor-merging crises in the gaps) in the rational approximation of level k, and ε^* denotes the quasiperiodic limit.

tractor in Fig. 2(c) becomes nonchaotic and resembles the single-band SNA in Fig. 1(c), although the level $k = 8$ is low. By increasing the level of the rational approximation to $k = 18$, we study the band-merging transition of the two-band torus, and found that the threshold value ε_k^*, at which the phase-dependent saddle-node bifurcation of level k (inducing the attractor-merging crises in the gaps) occured, converged to the quasiperiodic limit ε^* (= 0.126 662 718) in an algebraic manner, $|\Delta\varepsilon_k| \sim F_k^{-\alpha}$, where $\Delta\varepsilon_k = \varepsilon_k^* - \varepsilon^*$ and $\alpha \simeq 2.0$, as shown in Fig. 2(e). As the level k of the rational approximation increases, the number of gaps, where phase-dependent attractor-merging crises occur, becomes larger, and eventually in the quasiperiodic limit, the rational approximation to the attractor has a dense set of gaps, filled by single-band intermittent chaotic attractors. Consequently, an intermittent single-band SNA, containing the ring-shaped unstable set, appears, as shown in Fig. 1(c). We also note that this band-merging transition results in a truncation of the torus-doubling cascade.

To confirm the above mechanism for the band-merging transition, we also study the Toda oscillator with an asymmetric exponential potential, which is quasiperiodically forced at two incommensurate frequencies [16]:

$$\ddot{x} + \gamma\dot{x} + e^x - 1 = a\cos\omega_1 t + \varepsilon\cos\omega_2 t, \qquad (2)$$

where γ is the damping coefficient, a and ε represent the amplitudes of the quasiperiodic forcing, and ω ($\equiv \omega_2/\omega_1$) is irrational. By making a normalization, $\omega_1 t \to 2\pi t$, Eq. (2) can be reduced to three first-order differential equations,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{2\pi}{\omega_1}\gamma y + \frac{4\pi^2}{\omega_1^2}(-e^x + 1 + a\cos 2\pi t + \varepsilon\cos 2\pi\theta), \\ \dot{\theta} &= \omega \pmod 1. \end{aligned} \qquad (3)$$

By stroboscopically sampling the orbit points (x_n, y_n, θ_n) at the discrete time n, we obtain the 3D Poincaré map P with a constant Jacobian determinant of $e^{-\gamma T_1}$, where $T_1 = 2\pi/\omega_1$. Here, we set ω to be the reciprocal of the golden mean [*i.e.*, $\omega = (\sqrt{5} - 1)/2$)] and investigate the band-merging route to intermittent SNAs in the 3D Poincaré map P for the case of $\gamma = 0.8$ and $\omega_1 = 2.0$. Fig. 3(a) shows a phase diagram in the $a - \varepsilon$ plane. As in the case of the quasiperiodically forced Hénon map, a band-merging transition from a two-band smooth torus to a single-band intermittent SNA occurs when passing the white solid curve along route A. As an example, we consider the case of $a = 27$. Fig. 3(b) shows a two-band smooth torus in the Poincaré map P for $\varepsilon = 0.21$. When passing a threshold value $\varepsilon = \varepsilon^*$ (= 0.242 953 437), such a two-band torus transforms into a single-band intermittent SNA (*e.g.*, see the newly born intermittent SNA with $\sigma_1 \simeq -0.051$ and $\delta \simeq 6.2$ in Fig. 3(c) for $\varepsilon = 0.244$).

Using the rational approximation of level 8, we explain the mechanism for the band-merging transition to intermittent SNAs occurring in Figs. 3(b)-3(c) for $a = 27$. In P^2 (*i.e.*, the second iterate of the Poincaré map P), there exists a pair of conjugate smooth tori denoted by black curves, as shown in Fig. 3(d) for $\varepsilon = 0.207$. We note that conjugate ring-shaped unstable sets, represented by gray curves, lie close to the conjugate smooth tori. For this case, the smooth unstable torus, denoted by a black dashed line, is inaccessible from the interiors of the basins of the conjugate smooth tori; hence, it cannot induce any band-merging transition. As ε is increased, the

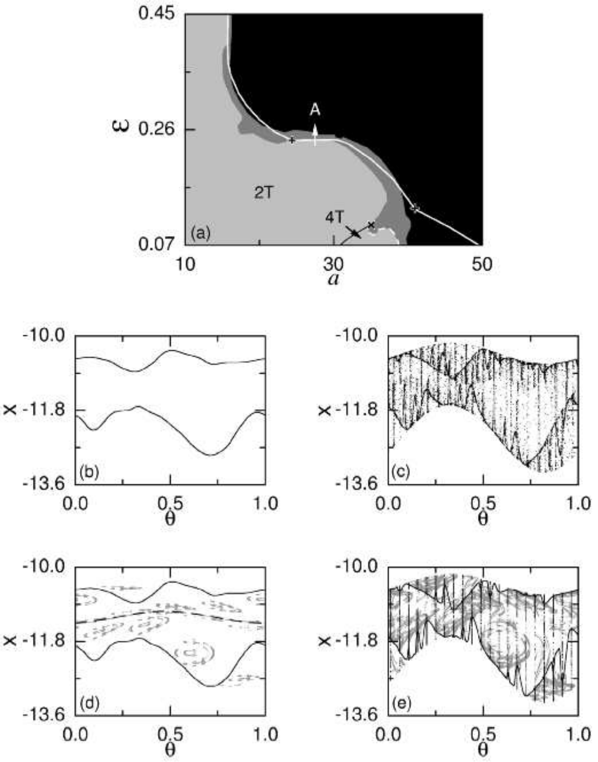

Fig. 3. (a) Phase diagram of the quasiperiodically forced Toda oscillator in the $a-\varepsilon$ plane for the case of $\gamma = 0.8$, $\omega_1 = 2.0$, and $\omega = (\sqrt{5}-1)/2$. Symbols and colors represent the same things as in Fig. 1(a). (b) and (c) Band-merging route to an intermittent SNA for $a = 27$ in the Poincaré map P. In (b) and (c), projections of the two-band smooth torus and the single-band intermittent SNA onto the $\theta - x$ plane are given for $\varepsilon = 0.21$ and 0.244, respectively. (d) and (e) Mechanism for the transition from a two-band smooth torus to a single-band intermittent SNA for $a = 27$ in the rational approximation of level 8. In the second iterate of the Poincaré map P (*i.e.*, P^2), projections of the conjugate tori (denoted by black solid curves), the ring-shaped unstable sets (represented by gray curves), and the unstable smooth torus (denoted by a dashed curve) onto the $\theta - x$ plane are given in (d) for $\varepsilon = 0.207$. Through collision between the conjugate tori and ring-shaped unstable sets, F_8 $(= 21)$ "gaps," filled by single-band intermittent chaotic attractors denoted by black dots, appear in the whole range of θ, as shown in (e) for $\varepsilon = 0.2409$. In (e), projections of the attractors (denoted by black dots) and the ring-shaped unstable sets (represented by gray curves) are plotted in the Poincaré map P.

conjugate tori and the ring-shaped unstable sets become closer. Eventually, when passing the threshold value ε_8^* $(= 0.240\,831\,592)$, a pair of phase-dependent saddle-node bifurcations occurs through collision between the conjugate tori and the ring-shaped unstable sets. Then, F_8 $(= 21)$ "gaps" without F_8-periodic attractors appear over the entire range of θ, as shown in Fig. 3(e) for $\varepsilon = 0.2409$. In these gaps, single-band intermittent chaotic attractors (denoted by black dots) appear. Thus, the rational approximation to the whole attractor in the Poincaré map P becomes composed of the union of the two-band periodic component and the single-band intermittent chaotic component. This partially-merged 8th rational approximation to the attractor in Fig. 3(e) becomes nonchaotic because $\langle\sigma_1\rangle \simeq -0.112$ and resembles the single-band intermittent SNA in Fig. 3(c), although the level $k = 8$ is low. Increasing the level of the rational approximation to $k = 18$, we obtain the threshold value ε_k^*, at which the phase-dependent saddle-node bifurcation of level k (mediating the attractor-merging crises in the gaps) occurs. As the level k is increased, the sequence $\{\varepsilon_k^*\}$ is found to converge to the quasiperiodic limit ε^* $(= 0.242\,953\,437)$ in an algebraic manner, $|\Delta\varepsilon_k| \sim F_k^{-\alpha}$, where $\Delta\varepsilon_k = \varepsilon_k^* - \varepsilon^*$ and $\alpha \simeq 2.0$. In the quasiperiodic limit $k \to \infty$, the rational approximation to the attractor has a dense set of gaps, filled by single-band intermittent chaotic attractors. As a result, an intermittent single-band SNA, containing the ring-shaped unstable set, appears, as shown in Fig. 3(c). In addition to the birth of SNAs, such a band-merging transition induces a truncation of the torus-doubling sequence.

III. SUMMARY

Using rational approximations to quasiperiodic forcing, we have investigated the dynamical mechanism for the band-merging route to intermittent SNAs in the quasiperiodically forced Hénon map. We have shown a two-band smooth torus transforms into a single-band intermittent SNA through a collision with a ring-shaped unstable set which has no counterpart in the unforced case. As a result, the torus-doubling cascade is truncated through this band-merging transition. The mechanism for this band-merging transition is also confirmed in the quasiperiodically forced Toda oscillator. Since the Hénon map and Toda oscillator are representative models for period-doubling systems, we believe that such a band-merging route to intermittent SNAs may occur in typical quasiperiodically forced period-doubling systems.

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