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Mechanism for New Boundary Crises in Quasiperiodically Forced Systems

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As a representative model for the Poincaré map of quasiperiodically forced oscillators, we consider the quasiperiodically forced Hénon map and investigate the mechanism for boundary crises. Using rational approximations to quasiperiodic forcing, we show that a new type of boundary crisis occurs for a nonchaotic attractor (smooth torus or strange nonchaotic attractor), as well as a chaotic attractor, through a collision with an invariant "ring-shaped" unstable set which has no counterpart in the unforced case. This new boundary crisis is in contrast to the "standard" boundary crisis that occurs via a collision with smooth unstable torus.

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Dynamical transitions of attractors occurring as the system parameters are changed have received much attention. Particularly, abrupt qualitative changes in the attractor are of great interest. Such discontinuous sudden changes, called crises, were first extensively studied by Grebogi *et al.* [1], and two kinds of boundary and interior crises were discovered for the case of chaotic attractors. Here, we consider the boundary crisis (BC), through which a chaotic attractor is suddenly destroyed when it collides with an unstable orbit on its basin boundary. Such a boundary crisis has been well investigated both theoretically [2] and experimentally [3] in periodically forced systems.

In this paper, we are interested in the BC in quasiperiodically forced systems driven at two incommensurate frequencies. These dynamical systems have attracted much attention because of the typical appearance of strange nonchaotic attractors which are strange (fractal), but nonchaotic (no positive Lyapunov exponent) [4]. Dynamical behaviors of quasiperiodically forced systems have been extensively investigated both theoretically [5–14] and experimentally [15]. In a recent work [11], a new type of BC was numerically observed to occur when the smooth unstable torus is inaccessible from the interior of the basin of the attractor due to the basin boundary metamorphosis [16]. However, the unstable orbit inducing such a BC was not located; thus, the mechanism for the new BC remains unclear.

We investigate the dynamical origin for the new BC in the quasiperiodically forced Hénon map M, often used as a representative model for the Poincaré map of quasiperiodically forced oscillators:

$$M : \begin{cases} x_{n+1} = a - x_n^2 + y_n + \varepsilon \cos 2\pi\theta_n, \\ y_{n+1} = bx_n, \\ \theta_{n+1} = \theta_n + \omega \pmod 1, \end{cases} \tag{1}$$

where a is the nonlinearity parameter of the unforced Hénon map, and ω and ε represent the frequency and amplitude of the quasiperiodic forcing, respectively. This quasiperiodically forced Hénon map M is invertible because it has a nonzero constant Jacobian determinant $-b$ whose magnitude is less than unity (*i.e.*, $b \neq 0$ and $-1 < b < 1$). Here we fix the value of the dissipation parameter b at $b = 0.05$.

We set the frequency ω to be the reciprocal of the golden mean, $\omega = (\sqrt{5} - 1)/2$. Then, using the rational approximation (RA) to this quasiperiodic forcing, we investigate the mechanism for the BC. For the inverse golden mean, its rational approximants are given by the ratios of the Fibonacci numbers, $\omega_k = F_{k-1}/F_k$, where the sequence of $\{F_k\}$ satisfies $F_{k+1} = F_k + F_{k-1}$ with $F_0 = 0$ and $F_1 = 1$. Instead of the quasiperiodically forced system, we study an infinite sequence of periodically forced systems with rational driving frequencies ω_k. We assume that the properties of the original system M may be obtained by taking the quasiperiodic limit $k \to \infty$.

Figure 1 shows a phase diagram in the $a - \varepsilon$ plane. Each phase is characterized by the (nontrivial) Lyapunov exponents, σ_1 and σ_2 ($\leq \sigma_1$), associated with the dynamics of the variables x and y (besides the zero exponent connected to the phase variable θ of the quasiperiodic forcing), as well as the phase sensitivity exponent δ. The exponent δ measures the sensitivity with respect to the phase of the quasiperiodic forcing and characterizes the strangeness of an attractor [8]. A smooth torus has negative Lyapunov exponents ($\sigma_{1,2} < 0$) and has no phase

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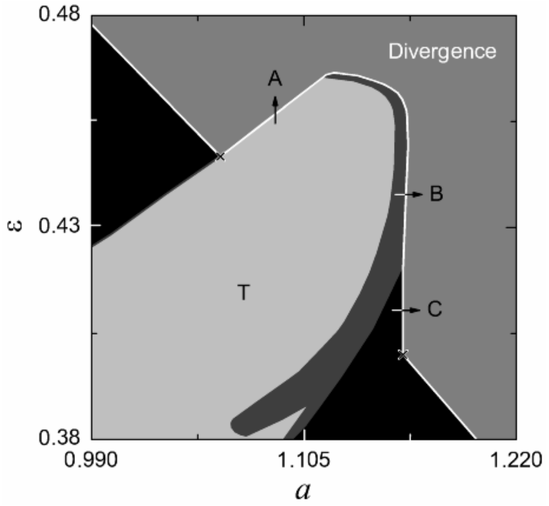

Fig. 1. Phase diagram in the $a-\varepsilon$ plane for the case of $b = 0.05$ and $\omega = (\sqrt{5}-1)/2$. Regular, chaotic, strange nonchaotic attractor, and divergence regions are shown in light gray, black, dark gray, and gray, respectively. A nonchaotic attractor [smooth torus (route A) or strange nonchaotic attractor (route B)], as well as a chaotic attractor (route C), is suddenly destroyed when passing the white solid curve. For other details, see the text.

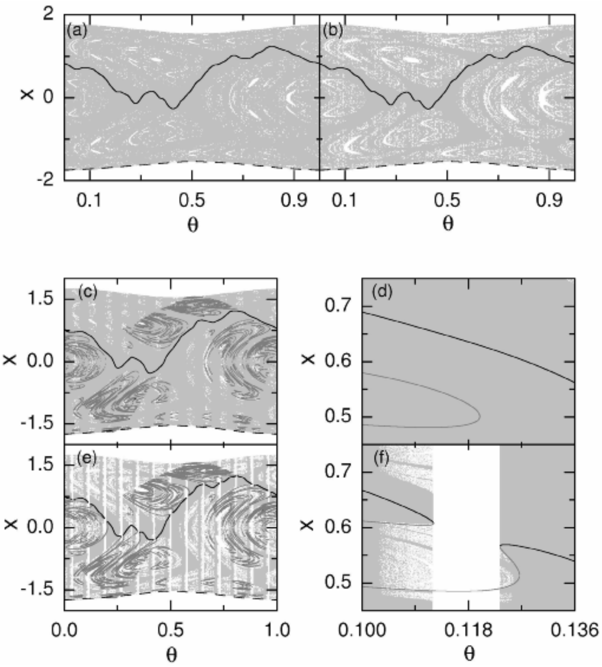

Fig. 2. In (a)-(f), projections of the attractor, the ring-shaped unstable set, and the smooth unstable torus onto the $\theta-x$ plane and the 2D slice with $y = 0$ of the basin are given. (a) and (b) BC of a smooth torus along the route A for $a = 1.09$. (a) Smooth torus (denoted by a black line) and its basin (shown in gray) for $\varepsilon = 0.435$. The unstable smooth torus (denoted by a dashed line) is not accessible from the interior of the basin of the smooth attracting torus because of the existence of holes (denoted by white dots). (b) Smooth torus and holes just before the BC for $\varepsilon = 0.44$. (c)-(f) Analysis of the mechanism for the BC of the smooth torus in the RA of level 7 for $a = 1.09$. Magnified views near $(\theta, x) = (0.118, 0.6)$ in (c) and (e) are given in (d) and (f), respectively. Here, the smooth torus whose basin is shown in gray, a ring-shaped unstable set, and holes are denoted by black, dark gray, and white dots, respectively. In (c) and (d) for $\varepsilon = 0.434$, a ring-shaped unstable set lies close to the smooth torus. For $\varepsilon = \varepsilon_7^*$ $(= 0.441\,629\,146)$, a BC occurs via phase-dependent saddle-node bifurcations between the smooth torus and the ring-shaped unstable set on the hole boundary. Then, F_7 $(= 13)$ "gaps," where divergence occurs, are formed, as shown in (e) for $\varepsilon = 0.442$, [*e.g.*, see a magnified gap in (f)].

sensitivity (*i.e.*, $\delta = 0$). Its region is denoted by T and is shown in light gray. On the other hand, a chaotic attractor has a positive Lyapunov exponent $\sigma_1 > 0$, and its region is shown in black. Between these regular and chaotic regions, strange nonchaotic attractors that have negative Lyapunov exponents $(\sigma_{1,2} < 0)$ and positive phase sensitivity exponents $(\delta > 0)$ exist in the regions shown in dark gray. Due to their high phase sensitivity, strange nonchaotic attractors are observed to have a strange fractal structure.

The attractors (smooth torus, strange nonchaotic attractor, and chaotic attractor) are abruptly destroyed via a BC inducing divergence (which occurs in the region shown in gray) when crossing the white solid curve in Fig. 1. We note that the BC curve loses its differentiability at the two vertices denoted by the crosses. A new type of BC occurs along the routes A, B, and C crossing the segment bounded by the two vertices. The new BC is in contrast to the standard BC which takes place for a chaotic attractor via a collision with the smooth unstable torus, which is developed from the unstable fixed point for the unforced case, on the remaining part of the BC curve. For the case of new BC, the smooth unstable torus becomes inaccessible from the interior of the basin of the attractor; hence, it cannot induce any BC. Through a collision with a new kind of ring-shaped unstable set, a nonchaotic attractor [smooth torus (route A) or strange nonchaotic attractor (route B)], as well as a chaotic attractor (route C), is suddenly destroyed. Using the RA, such a ring-shaped unstable set, which has no counterpart in the unforced case, was first discovered in our previous study on the intermittent route to strange nonchaotic attractors [13,14]. It appears via a phase-dependent saddle-node bifurcation. As the system parameters vary, both the sizes and the shapes of the rings constituting the unstable set are changed. Furthermore, as the level of the RA increases, the ring-shaped unstable set consists of a large number of rings; hence, it becomes a complicated unstable set. (For details on the structure and the evolution of the ring-shaped unstable set, refer to Fig. 2 of Ref. [13].)

We fix the value of a at $a = 1.09$ and investigate the BC of a smooth torus by varying ε along the route A. Figure 2(a) shows a smooth torus (denoted by a black curve) whose basin is shown in gray for $\varepsilon = 0.435$. We note that holes (shown in white), leading to divergence, exist inside the basin of the smooth attracting torus. Hence, the smooth unstable torus (denoted by the dashed line) is not accessible from the interior of the basin of the smooth attracting torus; hence, it cannot induce any BC. As the parameter ε increases, the smooth torus and holes become closer, as shown in Fig. 2(b) for $\varepsilon = 0.44$. Eventually, the smooth (attracting) torus is abruptly destroyed via a BC when it collides with the hole boundary for $\varepsilon = \varepsilon^*(= 0.457\,113\,401)$. Using the RA of level $k = 7$, we explain the mechanism for the BC of the smooth torus. Figure 2(c) shows the smooth torus (denoted by a black line), the ring-shaped unstable set (represented by dark gray dots), and holes (shown in white) for $\varepsilon = 0.434$. The RAs to the smooth torus and the ring-shaped unstable set are composed of stable and unstable orbits, respectively, with period F_7 (= 13). For this case, the ring-shaped unstable set is close to the smooth torus. However, it does not lie on any hole boundary [*e.g.*, see a magnified view in Fig. 2(d)]. As the parameter ε increases, the size of the holes increases, and new holes appear. Then, some part of the ring-shaped unstable set lies on a hole boundary. With further increases in ε, the smooth torus and the ring-shaped unstable set on the hole boundary become closer, and eventually, for $\varepsilon = \varepsilon_7^*$ (= 0.441 629 146), a phase-dependent saddle-node bifurcation occurs through a collision between the smooth torus and the ring-shaped unstable set. Then, "gaps," where the former attractor (*i.e.*, the stable F_7-periodic orbits) no longer exists and almost all trajectories go to the infinity, are formed, as shown in Fig. 2(e) for $\varepsilon = 0.442$ [*e.g.*, see a magnified gap in Fig. 2(f)]. As a result, a "partially-destroyed" torus with F_7 (= 13) gaps, where divergence occurs, is left. By increasing the level of the RA to $k = 18$, we study the BC of the smooth torus, and the threshold value ε_k^*, at which the phase-dependent saddle-node bifurcation of level k occurs, is found to converge to the quasiperiodic limit ε^* (= 0.457 113 401) in an algebraic manner: $|\Delta\varepsilon_k| \sim F_k^{-\alpha}$, where $\Delta\varepsilon_k = \varepsilon_k^* - \varepsilon^*$ and $\alpha \simeq 2.01$, as shown in Fig. 3. With an increase in the level k of the RA, the number of gaps where divergences take place becomes larger, and eventually in the quasiperiodic limit, a BC occurs in a dense set of gaps covering the whole θ-range. Consequently, the whole smooth torus disappears suddenly via a new type of BC when it collides with the ring-shaped unstable set. In a similar way, a strange nonchaotic attractor and a chaotic attractor are also destroyed suddenly through a collision with a ring-shaped unstable set when passing the BC curve along the routes B and C, respectively.

To sum up, by using the RAs to the quasiperiodic forcing, we have investigated the mechanism for the new BC in the quasiperiodically forced (invertible) Hénon map. The BC curve in the $a - \varepsilon$ plane loses its differentiability at two vertices. On the segment bounded by the two vertices, a new type of BC has been found to occur for a nonchaotic attractor (smooth torus or strange nonchaotic attractor), as well as a chaotic attractor, through a collision with a ring-shaped unstable set. This new BC is in contrast to the standard BC induced by the smooth unstable torus. We also note that the mechanism for the new BC is the same as that in the quasiperiodically forced (noninvertible) logistic map [17]. Since both the logistic and the Hénon maps are representative models for period-doubling systems, we believe that this kind of new BC might occur in typical quasiperiodically forced period-doubling systems.

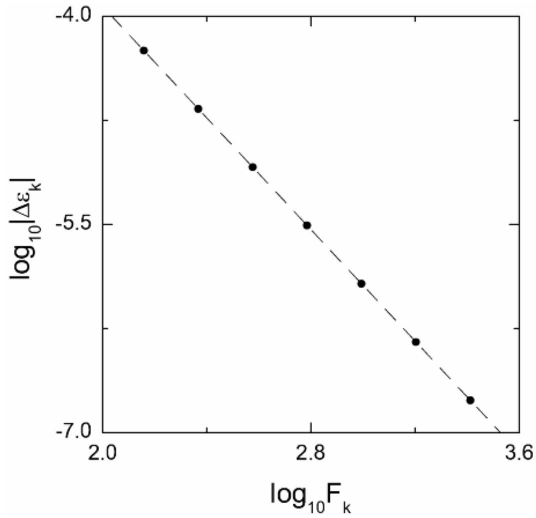

Fig. 3. Plot of $\log_{10}|\Delta\varepsilon_k^*|$ vs. $\log_{10} F_k$ for $k = 12, \ldots, 18$ [$\Delta\varepsilon_k^* = \varepsilon_k^* - \varepsilon^*$]. Here, ε_k^* (denoted by solid circles) represents the threshold value for the saddle-node bifurcation in the RA of level k, and ε^* denotes the quasiperiodic limit.

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