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# Renormalization analysis of two coupled maps

Sang-Yoon Kim
*Department of Physics, Kangwon National University, Chunchon, Kangwon-Do 200-701, South Korea*

and

Hyungtae Kook
*Department of Physics, Kyungwon University, Sungnam, Kyungki-Do 461-701, South Korea*

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The critical behavior of period-doubling bifurcations is studied in two coupled one-dimensional (1D) maps. We find three fixed maps of an approximate renormalization operator in the space of coupled maps. Each fixed map has a common relevant eigenvalue associated with scaling of the nonlinear parameter of the 1D map. However, relevant "coupling" eigenvalues associated with scaling of the coupling strength parameter vary depending on the fixed maps. The results of the renormalization analysis agree well with numerical results.

## 1. Introduction

Since the discovery of the universal scaling behavior of period doubling in low-dimensional dynamical systems [1,2], efforts have been made in studies of coupled systems to attempt to generalize to higher-dimensional systems [3–11]. Here we are concerned with the critical behavior of period doubling in two coupled one-dimensional (1D) maps.

The renormalization method has played a central role in the study of critical behavior of period doubling in low-dimensional maps; the fixed maps of the period-doubling renormalization operator for 1D [1] and two-dimensional (2D) area-preserving maps [2] have been found. In the case of four-dimensional (4D) volume-preserving maps (two coupled 2D area-preserving maps), Mao and Greene [8] found three fixed maps (truncated at quadratic terms) of an approximate renormalization operator, which is composed of squaring, truncating, and rescaling operators; hereafter we will refer to the approximate renormalization operator as the STR operator. Additional fixed maps (truncated at cubic terms) associated with "inverse" period doubling in the 4D volume-preserving maps have also been found [9]. In this paper, using the STR method, we study the critical behavior of period doubling in two coupled 1D maps.

In a recent numerical study on two coupled 1D maps [11], we found an infinite number of critical points. The structure of the critical set (set of critical points) depends on the nature of the coupling functions. In a linear coupling case in which the coupling function has a leading linear term, an infinite number of critical line segments, together with the previously known zero coupling point [3], constitute the critical set, whereas in the case of nonlinear coupling in which the leading term is nonlinear, the critical set consists of the only one critical line segment. The critical behavior depends on the position on the critical set. We found two kinds of new critical behavior at each critical line segment in the linear coupling case and one kind of new critical behavior at the interior points of the critical line in the nonlinear coupling case, in addition to the critical behavior at the zero coupling point.

These numerical results suggest that there exist three fixed maps of the period-doubling renormalization transformation in the space of coupled maps. A "zero-coupling" fixed map associated with the critical behavior at the zero coupling point has been found in ref. [3]. Using the STR method, we find two new fixed maps associated with the new critical behavior as well as the zero-coupling fixed map. All the three fixed maps have a common relevant eigenvalue associated with scaling of the nonlinearity parameter of the uncoupled 1D map. However, relevant "coupling" eigenvalues associated with scaling of the coupling strength parameter depend on the fixed maps. The values of the relevant eigenvalues are close to those of the parameter scaling factors obtained numerically using the scaling matrix method [11].

## 2. Two coupled 1D maps

We consider a map $T$ consisting of two identical 1D maps coupled symmetrically,

$$T:\ x_{i+1}=f(x_i)+g(x_i,y_i)\,,\qquad y_{i+1}=f(y_i)+g(y_i,x_i)\,, \tag{1}$$

where the subscript $i$ denotes a discrete time, $f(x)$ is a 1D map with a quadratic extremum, and $g(x,y)$ is a coupling function. Here the coupling function $g$ obeys the condition

$$g(x,x)=0 \quad \text{for any } x\,. \tag{2}$$

It follows from condition (2) that the partial derivatives of $g$ at $y=x$ satisfy a "sum rule",

$$\sum_{i=1}^{2} g_i(x,x)=0\,, \tag{3}$$

where the subscript $i$ of $g$ denotes the partial derivative of $g$ with respect to the $i$th argument.

Map (1) is invariant under the exchange of coordinates, $x\leftrightarrow y$. The set of points, which are invariant under the exchange of coordinates, forms a symmetry line, $y=x$. If an orbit lies on the symmetry line, then it is called an "in-phase" orbit; otherwise it is called an "out-of-phase" orbit. Here we study only in-phase orbits ($x_i=y_i$ for all $i$).

Let us introduce new coordinates, $X$ and $Y$,

$$X=\tfrac{1}{2}(x+y),\quad Y=\tfrac{1}{2}(x-y)\,. \tag{4}$$

Then the map $T$ of eq. (1) becomes

$$\begin{aligned} X_{i+1}&=F(X_i,Y_i)\\ &=\tfrac{1}{2}[f(X_i+Y_i)+f(X_i-Y_i)]\\ &\quad+\tfrac{1}{2}[g(X_i+Y_i,X_i-Y_i)+g(X_i-Y_i,X_i+Y_i)]\,,\\ Y_{i+1}&=G(X_i,Y_i)\\ &=\tfrac{1}{2}[f(X_i+Y_i)-f(X_i-Y_i)]\\ &\quad+\tfrac{1}{2}[g(X_i+Y_i,X_i-Y_i)-g(X_i-Y_i,X_i+Y_i)]\,. \end{aligned} \tag{5}$$

This map is invariant under the reflection, $Y\to -Y$, and hence, the symmetry line becomes $Y=0$. Then the in-phase orbit of the old map (1) becomes the orbit of this new map with $Y=0$. Moreover, since $g(X,X)=0$ for any $X$, the coordinate $X$ of the in-phase orbit satisfies the uncoupled 1D map, $X_{i+1}=f(X_i)$.

Linear stability of an in-phase orbit of period $p$ is determined from the Jacobian matrix $M$ of $T^p$, which is the $p$-product of the linearized map DT of map (5) along the orbit

$$M=\prod_{i=1}^{p} \mathrm{DT}(X_i,0) =\prod_{i=1}^{p}\begin{pmatrix} f'(X_i) & 0\\ 0 & f'(X_i)-2g_2(X_i,X_i)\end{pmatrix}, \tag{6}$$

where $f'(X)=\mathrm{d}f(X)/\mathrm{d}X$ and the sum rule (3) is used. The eigenvalues of $M$, called the stability multipliers of the orbit, are

$$\lambda_1=\prod_{i=1}^{p} f'(X_i)\,,\qquad \lambda_2=\prod_{i=1}^{p}[f'(X_i)-2g_2(X_i,X_i)]\,. \tag{7}$$

Note that $\lambda_1$ is just the stability multiplier of the orbit of the 1D map and coupling affects only $\lambda_2$. The in-phase orbit is stable only when the moduli of both multipliers are less than unity, i.e., $|\lambda_i|<1$ for $i=1$ and 2.

When its first stability multiplier $\lambda_1$ of an in-phase orbit passes through $-1$, this orbit loses its stability via "in-phase" period-doubling bifurcation, giving rise to the birth of the period-doubled in-phase orbit. The successive in-phase period-doubling bifurcations complete an infinite sequence. Unlike the 1D map case, there exists an infinite number of critical points in the space of the nonlinearity parameter and the coupling parameter. The structure of the critical set (set of critical points) depends on the nature of the coupling functions. We found three (two) kinds of critical behaviors in the linear (nonlinear) coupling case, depending on the position on the critical set. For details of the critical sets and the critical behavior refer to ref. [11].

### 3. Three fixed maps of the STR operator

The STR operator, which includes a truncation, is an approximation of the renormalization operator in the full function space of coupled maps. In the following procedure, the operation can be naturally represented by a transformation of parameters which correspond to the coefficients of truncated polynomials of coupled maps. The first step is to truncate map (5) at its quadratic terms, and then we obtain

$$T_{\boldsymbol{P}}:\ X_{i+1}=A/C+BX_i+CX_i^2+FY_i^2\,,$$
$$Y_{i+1}=DY_i+EX_iY_i\,, \tag{8}$$

which is a six-parameter family of coupled maps. Other terms do not appear since $F(X,Y)$ is even and $G(X,Y)$ is odd in $Y$ in eq. (5). Here $\boldsymbol{P}$ stands for the six parameters, i.e., $\boldsymbol{P}=(A,B,C,D,E,F)$. The construction of eq. (8) corresponds to a truncation of the infinite dimensional space of coupled maps to a six-dimensional space. The parameters, $A$, $B$, $C$, $D$, $E$, and $F$ can be regarded as the coordinates of the truncated space.

We look for fixed maps of the renormalization operator $\mathscr{R}$ in the truncated six-dimensional space of coupled maps,

$$\mathscr{R}(T)\equiv \Lambda T^2\Lambda^{-1}\,, \tag{9}$$

where $T$ and $\mathscr{R}(T)$ are within the truncated space, and $\Lambda$ is a rescaling operator. That is, the approximate renormalization operator $\mathscr{R}$ is composed of squaring ($T^2$), truncating (at quadratic terms), and rescaling ($\Lambda$) operators. Thus eq. (9) becomes the definition of the STR operator.

The old coordinates $x$ and $y$ of in-phase orbits scale with the 1D orbital scaling factor $\alpha$, since the location of in-phase orbits is determined from the 1D map. Therefore the rescaling operator $\Lambda$ in this coordinate system is

$$\Lambda=\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}. \tag{10}$$

$\Lambda$ is invariant under any linear coordinate changes, and hence, it is the rescaling operator also in the new coordinates $(X, Y)$.

The operation $\mathscr{R}$ with $\Lambda$ of eq. (10) in the truncated space can be represented by a transformation of parameters, i.e., a map from $\boldsymbol{P}\equiv(A,B,C,D,E,F)$ to $\boldsymbol{P}'\equiv(A',B',C',D',E',F')$,

$$\frac{A'}{C'}=\alpha\frac{A}{C}(1+B+A)\,, \tag{11a}$$

$$B'=B(B+2A)\,, \tag{11b}$$

$$C'=\frac{C}{\alpha}(B+B^2+2A)\,, \tag{11c}$$

$$D'=D(D+AE/C)\,, \tag{11d}$$

$$E'=\frac{E}{\alpha}(BD+D+AE/C)\,, \tag{11e}$$

$$F'=\frac{F}{\alpha}(2A+B+D^2)\,. \tag{11f}$$

The fixed point $\boldsymbol{P}^*$ of the map can be determined by solving $\boldsymbol{P}=\boldsymbol{P}'$. The parameter $C$ sets only the scale in $X$ and thus is arbitrary. We now fix the scale in $X$ by setting $C=1$. Then, we have, from eqs. (11a)–(11f), six equations for the six unknowns, $A$, $B$, $\alpha$, $D$, $E$, and $F$. We find three solutions for $\boldsymbol{P}^*$ associated with the period-doubling bifurcation, as will be seen below. Map (8) with a solution $\boldsymbol{P}^*$ ($T_{\boldsymbol{P}^*}$) is the fixed map of the STR operator; for brevity, $T_{\boldsymbol{P}^*}$ will be denoted a $T^*$.

We first note that eqs. (11a)–(11c) are for the unknowns $A$, $B$, and $\alpha$ only. We find four solutions for $A$, $B$, and $\alpha$,

$$A=\tfrac{1}{2}\alpha,\quad B=0,\quad \alpha=-1-\sqrt{3}=-2.732...\,, \tag{12a}$$

$$A=\tfrac{1}{2}\alpha,\quad B=0,\quad \alpha=-1+\sqrt{3}=0.732...\,, \tag{12b}$$

$$A=0,\quad B=1,\quad \alpha=2\,, \tag{12c}$$

$$A=\tfrac{1}{2}(1-B),$$
$$B=(-1+\sqrt{\tfrac{19}{27}})^{1/3}-(1+\sqrt{\tfrac{19}{27}})^{1/3}-1\,,$$
$$\alpha=1+B^2=8.668...\,. \tag{12d}$$

We compare the values of $\alpha$ with the numerically obtained value ($\alpha=-2.502...$) [1], and choose only the first case as the solution associated with the period-doubling bifurcation. Substituting the values of $A$, $B$, and $\alpha$ of eq. (12a) into eqs. (11d)–(11f), we obtain three solutions for $D$, $E$, and $F$,

$$D=0,\quad E=2,\quad F\text{: arbitrary real number}\,, \tag{13a}$$

$$D=0,\quad E=0,\quad F\text{: arbitrary real number}\,, \tag{13b}$$

$$D=1,\quad E=0,\quad F=0\,. \tag{13c}$$

All these three solutions are associated with the critical behavior of period doubling in coupled maps, as will be seen below. Hereafter, we will call each map from the top as $Z$, $I$, and $E$ map, respectively, as listed in table 1.

Consider an infinitesimal perturbation $\epsilon\delta\boldsymbol{P}$ ($\epsilon$ small) to a fixed point, $\boldsymbol{P}^*=(A^*, ..., F^*)$, of the transformation of parameters (11a)–(11f). Linearizing the transformation at $\boldsymbol{P}^*$, we obtain the equation for the evolution of $\delta\boldsymbol{P}$,

$$\delta\boldsymbol{P}'=J\delta\boldsymbol{P}\,, \tag{14}$$

where $J$ is the Jacobian matrix of the transformation at $\boldsymbol{P}^*$, i.e.,

$$J\equiv \left.\frac{\partial\boldsymbol{P}'}{\partial\boldsymbol{P}}\right|_{\boldsymbol{P}^*}\,. \tag{15}$$

A perturbation $\delta\boldsymbol{P}$ is said to be an eigenperturbation with eigenvalue $\lambda$ if

$$\delta\boldsymbol{P}'=\lambda\delta\boldsymbol{P}\,. \tag{16}$$

The eigenvalue $\lambda$ can be determined from the characteristic equation of the Jacobian matrix $J$,

$$\mathrm{Det}(J-\lambda I)=0\,, \tag{17}$$

where $I$ is the $6\times6$ unit matrix. Let us denote the eigenvector of $J$ with eigenvalue $\lambda$ as $\boldsymbol{V}=(V_1, ..., V_6)$. Then the infinitesimal eigenperturbation $\epsilon\boldsymbol{V}$ to $\boldsymbol{P}^*$ generates an infinitesimal eigenperturbation $\epsilon\delta T$ to the fixed map $T^*$ with the solution $\boldsymbol{P}^*$,

$$\delta T(X, Y)=(\delta F(X, Y), \delta G(X, Y))\,, \tag{18}$$

where

$$\delta F(X, Y)=\frac{1}{C^*}\left(V_1-\frac{A^*}{C^*}V_3\right)+V_2X+V_3X^2+V_6Y^2\,,$$

$$\delta G(X, Y)=V_4Y+V_5XY\,. \tag{19}$$

The evolution of a map $T$ ($=T^*+\epsilon\delta T$) close to a fixed map $T^*$ under the renormalization transformation $\mathscr{R}$ can be written as

$$\mathscr{R}(T^*+\epsilon\delta T)=T^*+\epsilon\delta T'+O(\epsilon^2)\,. \tag{20}$$

If $\delta T$ is an eigenperturbation (18) with eigenvalue $\lambda$, then the resulting perturbation $\delta T'$ in $\mathscr{R}(T)$ becomes $\delta T'=\lambda\delta T$.

The $6\times6$ Jacobian matrix $J$ decomposes into smaller blocks: two $1\times1$ blocks and two $2\times2$ blocks. For the $1\times1$ blocks, eigenvalues are just the entities of the blocks, that is,

$$\lambda_1=\left.\frac{\partial C'}{\partial C}\right|_{\boldsymbol{P}^*}=1,\quad \lambda_2=\left.\frac{\partial F'}{\partial F}\right|_{\boldsymbol{P}^*}=1+\frac{D^2}{\alpha}\,. \tag{21}$$

Here $\lambda_1$ is an eigenvalue associated with a scale change in $X$. The eigenvalue $\lambda_2$ is also associated with a scale change in $Y$ in the case $D=0$; this case corresponds to the $Z$ and $I$ maps in table 1, and hence, $F$ is arbitrary. However, in case $D=1$, corresponding to the $E$ map, $\lambda_2$ becomes an irrelevant eigenvalue whose modulus is less than unity. Note that the $E$ map is invariant under a scale change in $Y$ since $F=0$.

The remaining four eigenvalues are those of the following $2\times2$ blocks,

Table 1
Fixed point $\boldsymbol{P}^*$ (associated with the period doubling bifurcation) of eqs. (11a)–(11f) and the rescaling factor $\alpha$. We have fixed the scale in $X$ by setting $C=1$.

| Fixed map | $A$ | $B$ | $\alpha$ | $D$ | $E$ | $F$ |
|---|---|---|---|---|---|---|
| $Z$ | $\frac{1}{2}\alpha$ | 0 | $-1-\sqrt{3}$ | 0 | 2 | arbitrary |
| $I$ | $\frac{1}{2}\alpha$ | 0 | $-1-\sqrt{3}$ | 0 | 0 | arbitrary |
| $E$ | $\frac{1}{2}\alpha$ | 0 | $-1-\sqrt{3}$ | 1 | 0 | 0 |

$$M_1 = \left.\frac{\partial(A', B')}{\partial(A, B)}\right|_{P^*} = \left.\begin{pmatrix} \alpha(1+2A+B)+\frac{2A}{\alpha} & \alpha A+\frac{A}{\alpha}(1+2B) \\ 2B & 2(A+B) \end{pmatrix}\right|_{P^*}, \tag{22a}$$

$$M_2 = \left.\frac{\partial(D', E')}{\partial(D, E)}\right|_{P^*} = \left.\begin{pmatrix} 2D+AE & AD \\ \frac{E}{\alpha}(1+B) & \frac{D}{\alpha}(1+B)+2\frac{E}{\alpha}A \end{pmatrix}\right|_{P^*}. \tag{22b}$$

The two eigenvalues of $M_i$ ($i=1,2$) are called $\delta_i$ and $\delta'_i$ and are listed in table 2.

The three fixed maps have common eigenvalues, $\delta_1$ and $\delta'_1$, of $M_1$. These eigenvalues correspond to those of the 1D map. Compared with the numerical value ($\delta=4.669...$) [1] of the relevant eigenvalue $\delta$ associated with scaling of the nonlinearity parameter of the 1D map, the value of $\delta_1$ ($=3-\alpha$) is not bad for a first order approximation (truncating at quadratic terms) [#1]. The second eigenvalue, $\delta'_1=\alpha$ ($=-1-\sqrt{3}$), is associated with a shift of the $X$ coordinate.

The eigenvalues of $M_2$, $\delta_2$ and $\delta'_2$, are associated with coupling. These eigenvalues will be referred to as "coupling" eigenvalues. As shown in table 2, the $Z$ map has two relevant coupling eigenvalues, $\delta_2=\alpha$ and $\delta'_2=2$. The eigenperturbations (18) with these eigenvalues are as follows,

[#1] In the 1D map case, the accuracy of the Feigenbaum constants $\delta$ and $\alpha$ is improved in the second order approximation, as shown in table 2 of ref. [12].

$$\delta_2=\alpha=-1-\sqrt{3},$$

$$\delta F(X, Y)=0, \quad \delta G(X, Y)=Y-\frac{1}{2\alpha-1}XY. \tag{23a}$$

$$\delta'_2=2, \quad \delta F(X, Y)=0, \quad \delta G(X, Y)=XY. \tag{23b}$$

Unlike the case of the $Z$ map, the number of relevant coupling eigenvalues for the $I$ and $E$ maps is zero and one, respectively (see table 2). For the $I$ map, $M_2$ becomes a null matrix, and hence, there exists no coupling eigenvalue. Therefore it has only one relevant non-coordinate eigenvalue $\delta_1$, like the 1D map case. The $E$ map has one relevant coupling eigenvalue, $\delta_2=2$, and the corresponding eigenperturbation is

$$\delta F(X, Y)=0, \quad \delta G(X, Y)=Y. \tag{24}$$

In order to compare the results of the above renormalization analysis with those of our previous numerical study [11], we first make the scale changes $X\to AX$ and $Y\to AY$ in all the three fixed maps. Second, we fix the scale in $Y$ in the $Z$ and $I$ maps by setting $F=1$ (i.e., make a scale change $Y\to Y/\sqrt{F}$). Then the form of the fixed maps becomes

$$X_{i+1}=1+AX_i^2+GY_i^2,$$
$$Y_{i+1}=DY_i+EAX_iY_i, \tag{25}$$

where $G=A$ for the $Z$ and $I$ maps, and $G=0$ for the $E$ map.

The $Z$ map can be transformed back into the form,

$$x_{i+1}=1+Ax_i^2, \quad y_{i+1}=1+Ay_i^2. \tag{26}$$

with $x=X+Y$ and $y=X-Y$. Obviously, map (26) consist of two uncoupled 1D fixed maps (truncated at quadratic terms). Therefore the $Z$ map is associated with the critical behavior at the zero-coupling point [3]. The $Z$ map has two relevant coupling eigenvalues, $\delta_2$ and $\delta'_2$ (see table 2). In the coordinates

Table 2
Some eigenvalues, $\delta_1$, $\delta'_1$, $\delta_2$, and $\delta'_2$ of a fixed map $T^*$ of the renormalization operator are shown. In the last two columns, the values of the stability multipliers of a fixed point of the fixed map are listed, and $\alpha=-1-\sqrt{3}$.

| Fixed map | $\delta_1$ | $\delta'_1$ | $\delta_2$ | $\delta'_2$ | $\lambda_1^*$ | $\lambda_2^*$ |
|---|---|---|---|---|---|---|
| $Z$ | $3-\alpha$ | $\alpha$ | $\alpha$ | 2 | $1-\sqrt{1-2\alpha}$ | $1-\sqrt{1-2\alpha}$ |
| $I$ | $3-\alpha$ | $\alpha$ | nonexistent | nonexistent | $1-\sqrt{1-2\alpha}$ | 0 |
| $E$ | $3-\alpha$ | $\alpha$ | 2 | $\alpha^{-1}$ | $1-\sqrt{1-2\alpha}$ | 1 |

$(x, y)$, the eigenperturbations (23a) and (23b) become,

$$\delta_2 = \alpha = -1-\sqrt{3}\,,$$

$$x_{i+1} = \delta g(x_i, y_i)$$

$$= \tfrac{1}{2}(x_i - y_i)\left(1 - \frac{A}{2(2\alpha-1)}(x_i+y_i)\right),$$

$$y_{i+1} = \delta g(y_i, x_i)\,. \tag{27}$$

$$\delta_2' = 2\,,$$

$$x_{i+1} = \delta g(x_i, y_i) = \tfrac{1}{4}A(x_i - y_i)(x_i + y_i)\,,$$

$$y_{i+1} = \delta g(y_i, x_i)\,. \tag{28}$$

Note that the leading term of eigenperturbation (27) is linear, whereas that of eigenperturbation (28) is nonlinear. Therefore the critical behavior at the zero coupling point depends on the nature of the coupling, i.e., the leading term of the coupling function. The values of $\delta_2$ and $\delta_2'$ are close to the numerical values of the coupling-parameter scaling factor $\gamma_2$ [11] at the zero coupling point in the linear ($\gamma_2 = -2.502...$) and nonlinear ($\gamma_2 = 1.999...$) coupling cases, respectively.

At a critical point, stability multipliers $\lambda_{1,n}$ and $\lambda_{2,n}$ of an in-phase orbit of period $2^n$ converge to the critical stability multipliers, $\lambda_1^*$ and $\lambda_2^*$ as $n \to \infty$ [11]. In addition to the coupling eigenvalues, we also obtain these critical stability multipliers. The invariance of a fixed map $T^*$ under the renormalization transformation $\mathscr{R}$ implies that, if $T^*$ has a periodic point $(x, y)$ with period $2^n$, then $A^{-1}(x, y)$ is a periodic point of $T^*$ with period $2^{n+1}$. Since rescaling does not affect the stability multipliers, all in-phase orbits of period $2^n$ ($n=0, 1, ...$) have the same stability multipliers, which are just the critical stability multipliers, $\lambda_1^*$ and $\lambda_2^*$. Therefore the critical stability multipliers have the values of the stability multipliers of a fixed point $(\hat{X}, 0)$ of the fixed map (25),

$$\lambda_1^* = 2A\hat{X}, \quad \lambda_2^* = D + EA\hat{X}\,, \tag{29}$$

where $\hat{X} = (1-\sqrt{1-2\alpha})/\alpha$. The three fixed maps have a common critical stability multiplier, $\lambda_1^* = 1-\sqrt{1-2\alpha} = -1.542...$, which is close to the numerical value of the critical stability in the 1D map case, $\lambda^* = -1.601...$ [1]. However, $\lambda_2^*$ depends on the fixed maps. For the $Z$ map, $\lambda_2^*$ becomes the same as $\lambda_1^*$.

For the $I$ map, eq. (25) becomes

$$X_{i+1} = 1 + AX_i^2 + AY_i^2, \quad Y_{i+1} = 0\,. \tag{30}$$

The second critical stability multiplier for this case is $\lambda_2^* = 0$. As mentioned earlier, this fixed map with zero Jacobian determinant has no relevant coupling eigenvalue, and hence, it has only one relevant noncoordinate eigenvalue $\delta_1$, like the 1D map case. In ref. [11], we found that, at interior points of the critical line segments in the linear and nonlinear coupling cases, the second critical stability multiplier is zero and there exists no coupling-parameter scaling factor, i.e., the critical behavior at interior points is essentially the same as that in the 1D map case. The $I$ map, therefore, governs the critical behavior at interior points of critical line segments in the linear and nonlinear coupling cases.

For the $E$ map, eq. (25) becomes

$$X_{i+1} = 1 + AX_i^2, \quad Y_{i+1} = Y_i\,. \tag{31}$$

As shown in table 2, this fixed map has one relevant coupling eigenvalue $\delta_2 = 2$, which agrees well with the numerical value ($\gamma_2 = 1.999...$) [11] of the coupling-parameter scaling factor $\gamma_2$ at both ends of each critical line segment in the linear coupling case. The second critical stability multiplier is $\lambda_2^* = 1$, which also agrees well with the numerical value ($\lambda_2^* = 1.000...$) [11] of the second critical stability multiplier at both ends of each critical line segment in the linear coupling case. Therefore the $E$ map is associated with the critical behavior at both ends of each critical line segment in the linear coupling case.

## 4. Summary

Using an approximate renormalization method (truncating at quadratic terms), we study the critical behavior of period-doubling in two coupled 1D maps. We find two new fixed maps ($I$ and $E$ maps) as well as the previously found $Z$ map. The $I$ and $E$ maps are associated with the new critical behavior observed at interior points of each critical line segment in the linear and nonlinear coupling cases and at both ends of each critical line segment in the linear coupling case, respectively.

All fixed maps have a common relevant eigenvalue, associated with the scaling behavior of the nonlinearity parameter of the uncoupled 1D map. However, the number of relevant coupling eigenvalues for the $Z$, $E$, and $I$ maps is two, one, and zero, respectively. The values of the relevant coupling eigenvalues agree well with the numerical values of the coupling-parameter scaling factor. Also, the values of the stability multipliers of the fixed point of the fixed maps are close to the numerical values of the critical stability multipliers.


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