UNIVERSALITY OF $k\ 3^n$ AND $k\ 4^n$ BIFURCATIONS IN AREA-PRESERVING MAPS

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We study period-trebling and period-quadrupling bifurcations in two-dimensional reversible area-preserving maps Our numerical results show that there are unique universal limiting behaviors in each of the period-trebling and period-quadrupling sequences

The area-preserving maps possess a generic property of nonintegrable hamiltonian systems A dominant cause of the onset of chaotic motion is often attributed to the period-multiplying bifurcation phenomena The understanding of chaotic behavior of hamiltonian systems is important for the understanding of both statistical systems (completely stochastic systems) and partially regular systems such as plasmas in fusion devices and particles in accelerators

When the nonintegrable parameter varies, period-multiplying bifurcations occur by emitting stable and unstable periodic orbits nearby the original orbit Except the period-doubling bifurcations, the original orbit remains stable, while one of the daughter orbits is stable and the other is unstable In the vicinity of the stable orbit invariant circles (KAM circles) are formed, while in the vicinity of the unstable orbit a stochastic layer is formed Therefore the period-multiplying bifurcations are as important as the period-doubling bifurcations in the onset of chaotic motion

In the case of period-doubling bifurcations the universality of the bifurcation sequences has been studied extensively both for area-preserving and dissipative maps [1,2] Unlike the one-dimensional map, period-trebling and period-quadrupling sequences do exist in the area-preserving maps Therefore it is natural to ask if there are universal limiting behaviors in these sequences

In this work we study the period-trebling ($k\ 3^n$) and the period quadrupling ($k\ 4^n$) bifurcations in the 2D reversible area-preserving maps Except brief comments on the period-trebling bifurcation in refs [1] and [3], the present study is, to our knowledge, the first detailed work on these sequences In particular the $k\ 4^n$ bifurcation is never studied before As shown elsewhere [4], all quadratic maps are equivalent and we take the following form of the Henon quadratic map as a typical representative of the reversible 2D area-preserving map of T $x_{n+1} = -y_n + 2h(x_n), y_{n+1} = x_n$ with $h(x) = \frac{1}{2}(1 - ax^2)$ The two symmetry lines formed from the invariant points of the involutions of TS and S with $(TS)^2 = 1 = S^2$, where TS $x_{n+1} = -x_n + 2h(y_n), y_{n+1} = y_n$ and S $x_{n+1} = y_n, y_{n+1} = x_n$ are $x = h(y)$ of TS and $y = x$ of S When the residue R, defined as $R = \frac{1}{4}(2 - \mathrm{Tr}\, M)$ with M being the jacobian matrix of T^l about an orbit of period l, varies near $R = 3/4$ with $R = \sin^2(\pi\mu/\nu)$ of $\mu = 1, \nu = 3$, the three unstable points of period $3l$ move toward the stable point of period l as the R value approaches $R = 3/4$ from the lower side [5] At $R = 3/4$ they are

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absorbed by the stable point and as R increases three new unstable points are reemitted from the stable point [5]. As explained in ref. [1], when one periodic point $P_n(0)$ of the odd period l is on the symmetry line of S, $y = x$, then the $[(l+1)/2]$th point from that point is on the other symmetry line of TS, $x = h(y)$ ($P_n(S) = T^sP_n(0)$, where $P_n(0)$ is the initial point of the $3^n(=l)$-periodic orbit). Fig. 1 shows consecutively enlarged figures of the period-trebling bifurcations associated with two periodic points on S and TS. The center circles inside the triangle are the lower-order periodic points which are surrounded by the three stable points of the next-higher-order period-trebled orbit. At the parameter value where the $l = 3^n$, 3^{n+1}, and 3^{n+2} orbits are all stable, the consecutive triangles in fig. 1 are the enlarged figures near the area

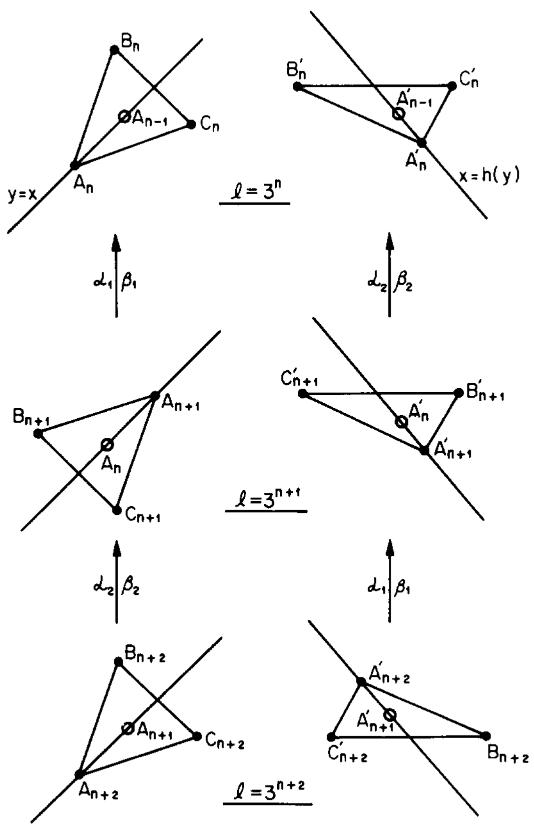

Fig. 1. Period-trebling cascades associated with two periodic points on S and TS. A_n, B_n, C_n, A'_n, B'_n and C'_n are elements of an l-periodic orbit corresponding to $P_n(0)$, $P_n(l/3)$, $P_n(2l/3)$, $P_n((l+1)/2)$, $P_n((l+1)/2 + l/3)$ and $P_n((l+1)/2 - l/3)$ of the text with $l = 3^n$ respectively.

around the previous low-order periodic orbits $P(0)$ (the A_n in fig. 1) and $P[(l+1)/2]$ (the A'_n in fig. 1) on S and TS. The two periodic points off the symmetry lines are reflection points of each other with respect to that symmetry line. As indicated in the figure with α_1 and α_2, the higher-order periodic point on the symmetry line appears either on the different side with respect to the center circle (α_1) or on the same side (α_2), thus flipping the triangle figure (α_1) or leaving the figure the same (α_2). However, these flipping (α_1) or leaving (α_2) occur at the same time, one on one symmetry line and the other on the other symmetry line. In the next enlarged figures α_1 and α_2 are interchanged and as a consequence the rescaling can be done at every other trebling, that is nine-tupling, rather than a single trebling. Therefore the complete rescaling factor on the symmetry line α is given by $\alpha = \alpha_1\alpha_2$. Similarly the scaling factor associated with off the symmetry line is $\beta = \beta_1\beta_2$. The third universal number δ is given by $\delta = \delta_1^2$ with $\delta_1 = \delta_2$. α_j, β_j, δ_j are the limiting values of $\alpha_n(j)$, $\beta_n(j)$, $\delta_n(j)$, $j = 1,2$ (table 1) when $n \to \infty$ is taken, $\delta_n(1) = \delta_n(2) = (a_{n-1} - a_n)/(a_n - a_{n+1})$. The values of a_n are those at which the stable periodic orbits of period $l = 3^n$ become unstable. We obtain the accumulation point $a_\infty = 1.181569424$. We may recall a crude approximation of Derrida and Pomeau [3] giving $a = 1.18362$. In table 1 the numerical values of $\alpha_n(j)$, $\beta_n(j)$, $\delta_n(j)$ are given up to $n = 7$ although periodic orbits are found up to $n = 8$. We conclude from these numerical studies that $\alpha_1 = -17.9$, $\alpha_2 = 2.45$, $\beta_1 = 6.02$, $\beta_2 = -31.0$ and $\delta_1 = \delta_2 = 20.2$. Thus $\alpha = \alpha_1\alpha_2 = -44.0$, $\beta = \beta_1\beta_2 = -187$ and $\delta = \delta_1^2 = (20.2)^2 = 408$. We studied two trebling bifurcations of $2 \cdot 3^n$ and $6 \cdot 3^n$ and find the same results for both cases, except that both triangles are on the same symmetry line, the former on the TS and the latter on the S symmetry. The latter case of the basic period 6 is the orbit bifurcated from the original orbit of period 3.

As for the period-quadrupling bifurcations, four unstable orbits are adsorbed and reemitted at $R = 1/2$, similar to the case of period-trebling. Fig. 2 shows that among the higher-order periodic points surrounding the low-order points on TS line of $x = h(y)$, either two or none are on the symmetry line. In the case of two points on the line, $P_n(0)$ and $P_n(l/2)$ (A_n and C_n of fig. 2) surround the center circle of $P_{n-1}(0)$ and the case of none on the line occurs in the surrounding

Table 1
Sequence of alternating period-trebling cascade (1 and 2 refer to alternating scaling factors) $\alpha_n(1) = \overline{(A_n - B_n)_S}/\overline{(A_{n+1} - B_{n+1})_S}$, $\alpha_n(2) = \overline{(A'_n - B'_n)_y}/\overline{(A'_{n+1} - B'_{n+1})_y}$, $\beta_n(1) = \overline{B_n - C_n}/\overline{B_{n+1} - C_{n+1}}$ and $\beta_n(2) = \overline{B'_n - C'_n}/\overline{B'_{n+1} - C'_{n+1}}$ are defined with respect to fig 1 $\overline{(A_n - B_n)_S}$ denotes the length of the projection of the line $A_n - B_n$ onto the S-symmetry line $\overline{(A'_n - B'_n)_y}$ denotes the length of the projection of the line $A'_n - B'_n$ onto the y-axis $\overline{B_n - C_n}$ and $\overline{B'_n - C'_n}$ denote respectively the length of the line $B_n - C_n$ and the line $B'_n - C'_n$ $\delta_n(1) = \delta_n(2) = (a_{n-1} - a_n)/(a_n - a_{n+1})$ where the a_n are the parameter values at which $l(= 3^n)$-periodic orbits turn unstable with hyperbolic reflection

n	$\delta_n(1)$	$\alpha_n(1)$	$\alpha_n(2)$	$\beta_n(1)$	$\beta_n(2)$
2	20 2460846	−17 9480767	2 38276536	6 04323293	−31 6525482
3	20 3215360	2 46433440	−17 9670829	−31 1407430	6 02062133
4	20 1880509	−17 9218285	2 45248666	6 02337526	−31 0473923
5	20 1878211	2 45392197	−17 9274593	−31 0341416	6 01698339
6	20 1848093	−17 8917711	2 45339397	6 01703527	−31 0329655
7	20 1848339	2 45819187	−17 9262130	−31 0293196	6 01711290

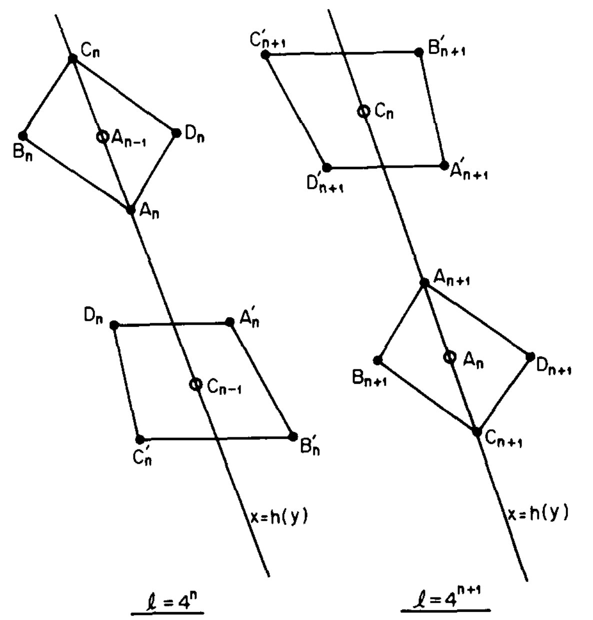

Fig 2 Period-quadrupling cascades associated with two periodic points on TS $A_n, B_n, C_n, D_n, A'_n, B'_n, C'_n$ and D'_n are elements of an l-periodic orbit corresponding to $P_n(0), P_n(l/4), P_n(l/2), P_n(3l/4), P_n(l/8), P_n(3l/8), P_n(5l/8)$ and $P_n(7l/8)$ of the text with $l = 4^n$ respectively

area of the center circle of $P_{n-1}(l/2)$ In the next enlargement of the figure, two points of $P_{n+1}(0)$ and $P_{n+1}(l/2)$ on the TS line surround $P_n(0)$ while all four points (A'_n, B'_n, C'_n, and D'_n of fig 2) around $P_n(l/2)$ are off the line If we define $\alpha_n(j)$, $\alpha_n(j)$ for j = 1,2,3 (table 2), in order to retain the information of the shape of the figure made out of high-order points we find from table 2 that all $\alpha_n(j)$ and $\beta_n(j)$ approach the same limits irrespective of j, contrary to the case of the trebling bifurcation That is we obtain $\alpha = -5\ 61$ and $\beta = 14\ 3$, respectively, as the $n \to \infty$ limits Also the same definition as before of $\delta_n = (a_{n-1} - a_n)/(a_n - a_{n+1})$ gives the limiting value $\delta = 24\ 5$ and we

Table 2
Period-quadrupling cascade sequences $\alpha_n(1) = \overline{(A_n - B_n)_y}/\overline{(A_{n+1} - B_{n+1})_y}$, $\alpha_n(2) = \overline{(B_n - C_n)_y}/\overline{(B_{n+1} - C_{n+1})_y}$, $\alpha_n(3) = \overline{(A'_n - B'_n)_y}/\overline{(A'_{n+1} - B'_{n+1})_y}$, $\beta_n(1) = \overline{B_n - D_n}/\overline{B_{n+1} - D_{n+1}}$, $\beta_n(2) = \overline{A'_n - D'_n}/\overline{A'_{n+1} - D'_{n+1}}$ and $\beta_n(3) = \overline{B'_n - C'_n}/\overline{B'_{n+1} - C'_{n+1}}$ are defined with respect to fig 2 $\overline{(A_n - B_n)_y}$, $\overline{(B_n - C_n)_y}$ and $\overline{(A'_n - B'_n)_y}$ denote the length of the projection onto the y-axis of the line $A_n - B_n$, the line $B_n - C_n$ and the line $A'_n - B'_n$ respectively $\overline{B_n - D_n}$, $\overline{A'_n - D'_n}$ and $\overline{B'_n - C'_n}$ denote respectively the length of the line $B_n - D_n$, the line $A'_n - D'_n$ and the line $B'_n - C'_n$ The δ_n are defined similar to the δ_n of table 1

n	δ_n	$\alpha_n(1)$	$\alpha_n(2)$	$\alpha_n(3)$	$\beta_n(1)$	$\beta_n(2)$	$\beta_n(3)$
2	23 434637	−5 4876778	−5 6671749	−4 5801677	14 602567	16 934541	16 083341
3	25 021692	−5 6124485	−5 5665575	−5 9910540	14 323708	13 490973	13 611233
4	24 454647	−5 6116719	−5 6304191	−5 4967372	14 297812	14 589097	14 587388
5	24 478091	−5 6178999	−5 6115809	−5 6416557	14 275449	14 209793	14 212675

obtain a_∞ = 0 1427235 The Universal constants α, β, and δ obtained from 4^n and 6 4^n bifurcations are shown to be the same The latter case of basic period 6 is the orbit bifurcated from the original orbit of period 3 so that the quadrilaterals discussed above are all on the S-symmetry line When we apply the renormalized scheme developed by Derrida and Pomeau [3] to the 4^n bifurcation, the recursion relation gives $\sqrt{1+a} - a = 8a' + 16a'^{3/2} - 1$ a' is the parameter value of T^4 which looks like T with parameter value a in the lowest-order approximation The accumulation point a_∞ is obtained as the fixed point of the recursion relation and the value is a_∞ = 0 1467, while the δ from $\delta = \mathrm{d}a/\mathrm{d}a'(a = a_\infty)$ is given by 24 7 Both values are to be compared with the numerical results a_∞ = 0 1427235 and δ = 24 5

In summary, we find numerically that period-trebling and period-quadrupling sequences have universal limiting behaviors and we have also confirmed our numerical results by a simple approximate renormalization method

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