# POWER SPECTRA OF HIGHER PERIOD MULTIPLINGS IN AREA-PRESERVING MAPS

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We have studied the power spectra of higher period $p$-multiplings for $p=3$, 4, 5 and 6 in area-preserving maps. The ratio of the successive average heights of the peaks of the spectrum for each $p$-multipling is found to approach a universal scaling limit that increases with $p$.

The study of periodic orbitals in Hamiltonian systems is relevant to a number of important problems such as plasma confinement in fusion, particle instability in beam–beam interactions, and celestial mechanics. A very useful approach to the study of Hamiltonian systems is the method of the Poincaré surface of section. Hamiltonian systems with $n$ degrees of freedom can be modeled by $2(n-1)$-dimensional symplectic maps. For Hamiltonian systems with two degrees of freedom, they are simply described by area-preserving maps.

To relate theory to experiment, one of the most directly accessible quantities is the power spectrum that contains information of the global scaling behavior. The power spectra of period doubling in one and two dimensions have been studied, and they are found to exhibit universal scaling properties [1,2]. In two dimensions, besides period doubling, higher period $p$-multiplings also occur [3,4]. The local scaling behaviors of higher period $p$-tuplings have been studied [5–8]. It is therefore interesting to study the power spectra of the higher period $p$-multiplings to examine their global scaling behavior. In this paper we have studied the power spectra of period multiplings for $p=3$, 4, 5 and 6.

The map was use in this study is the area-preserving Hénon map $T$:

$$T:\quad x' = -y + 2f_a(x)\,,\quad y' = x\,, \tag{1}$$

where $f_a(x) = \frac{1}{2}(1 - ax^2)$. The map can be written as the product of two involutions $S^2 = (TS)^2 = 1$, where

$$S:\quad x' = y\,,\quad y' = x\,, \tag{2a}$$

$$TS:\quad x' = -x + 2f_a(y)\,,\quad y' = y\,. \tag{2b}$$

This symmetry property greatly facilitates the search for periodic orbits [5].

To study the power spectrum, let us consider a periodic orbit $x_i$ with period $p^n$ (i.e., $x_{i+p^n} = x_i$). Its Fourier transform is given by

$$x(\omega) = \frac{1}{\sqrt{p^n}} \sum_{j=1}^{p^n} e^{ij\omega} x_j\,, \tag{3}$$

where $\omega=2\pi m/p^n$, $m=0, 1, ..., p^n-1$. The power spectrum is

$$P(\omega)=|x(\omega)|^2=\frac{1}{p^n}\sum_{j,n=1}^{p^n} e^{i\omega(j-m)}x_j x_m=\frac{1}{p^n}\left(\sum_{j=1}^{p^n} x_j^2+\sum_{j>m}(e^{i\omega(j-m)}+\text{c.c.})x_j x_m\right)$$

$$=\frac{1}{p^n}\left(\sum_{j=1}^{p^n} x_j^2+2\sum_{l=1}^{p^n-1}\sum_{m=1}^{p^n-l} x_m x_{m+l}\cos\omega l\right)=S(0)+2\sum_{l=1}^{p^n-1} S(l)\cos\omega l\,, \tag{4}$$

where

$$S(l)=\frac{1}{p}\sum_{j=1}^{p^n-l} x_j x_{j+l}\,. \tag{5}$$

As $n\to\infty$ and $l\ll p^n$, $S(l)$ tends to the autocorrelation function introduced in ref. [1]. In two dimensions, $x_j$ is a two-vector.

For a period-$p^n$ orbit, the power spectrum is discrete and the peaks of the spectrum are delta functions that appear at $\omega=2\pi m/p^n$, $m=0, 1, ..., p^n-1$. As a period-$p$ bifurcation occurs, new peaks will appear at $\omega=2\pi(pm-I)/p^{n+1}$, where $I=1, ..., p-1$, and $m=1, ..., p^n$. To classify the contributions of successive bifurcations in the power spectrum, we write

$$P(\omega)=P_{000}\delta(\omega)+\sum_{k=1}^{n}\sum_{I=1}^{p-1}\sum_{m=1}^{p^{k-1}} P_{kmI}\,\delta\!\left(\omega-\frac{2\pi(pm-I)}{p^k}\right), \tag{6}$$

where

$$P_{kmI}=S(0)+2\sum_{m=1}^{p^{k-1}} S(j)\cos\!\left(\frac{2\pi(pm-I)j}{p^k}\right) \tag{7}$$

is the height of each peak in the spectrum. The quantity of interest is the ratio of the successive average heights of the peaks. The average height of the peaks generated by the $k$th bifurcation in the power spectrum is defined as

$$\phi_p(k)=\frac{1}{p-1}\sum_{I=1}^{p-1}\frac{1}{p^{k-1}}\sum_{m=1}^{p^{k-1}} P_{kmI}\,. \tag{8}$$

Inserting eq. (7) into eq. (8), we get

$$\phi_p(k)=S(0)+2\sum_{j=1}^{p^n-1} S(j)\,\frac{1}{(p-1)p^{k-1}}\sum_{I=1}^{p-1}\sum_{m=1}^{p^{k-1}}\cos\!\left(\frac{2\pi(pm-I)j}{p^k}\right).$$

The sum can be easily evaluated:

$$\sum_{I=1}^{p-1}\sum_{m=1}^{p^{k-1}}\cos\!\left(\frac{2\pi(pm-I)j}{p^k}\right)=(p-1)p^{k-1}\,,\quad \text{if } j=p^k l \ (l \text{ an integer})$$

$$=-p^{k-1}\,,\quad \text{if } j=p^{k-1} \text{ and } l\neq pq \ (q \text{ and } l \text{ integers});$$

$$=0\,,\quad \text{otherwise.} \tag{10}$$

We have therefore

$$\phi_p(k)=S(0)+2\left(\sum_{l=1}^{p^{n-k}-1} S(p^k l)-\frac{1}{p-1}\sum_{l=1}^{p^{n-k}}\sum_{I=1}^{p-1} S(p^{k-1}(pl-I))\right),\quad \text{for } k<n\,,$$

$$=S(0)-\frac{2}{p-1}\sum_{I=1}^{p-1} S(p^{k-1}(p-I))\,,\qquad \text{for } k=n\,. \tag{11}$$

Table 1
The ratios $2\beta^{(p)}$ of the successive average heights of the power spectra of multipling. Column (a) is from the autocorrelation function and column (b) from FFT.

| $k$ | $p=2$ | | $p=3$ | | $p=4$ | | $p=5$ | | $p=6$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | (a) | (b) | (a) | (b) | (a) | (b) | (a) | (b) | (a) | (b) |
| 3 | 53.6 | 53.4 | 121.6 | 120.9 | 165.4 | 165.4 | 189.6 | 187.0 | 221.1 | 221.1 |
| 4 | 62.4 | 62.4 | 122.4 | 122.9 | 171.1 | 171.1 | 195.2 | 195.6 | 223.8 | 223.8 |
| 5 | 58.4 | 58.6 | 122.0 | 122.0 | 168.8 | 168.8 | 193.4 | 194.7 | 223.9 | 223.9 |
| 6 | 61.0 | 60.2 | 122.2 | 122.2 | 170.8 | 170.8 | 195.2 | 195.2 | | |
| 7 | | | 122.5 | 122.0 | | | | | | |
| 8 | | | 123.1 | 123.0 | | | | | | |
| avg. | | 60 | | 123 | | 170 | | 195 | | 224 |
| rate | | | | 2.05 | | 1.38 | | 1.15 | | 1.15 |

Asymptotically, the ratio of the successive average heights of the peaks

$$2\beta^{(p)}=\phi_p(k)/\phi_p(k+1) \qquad (12)$$

approaches a universal limit. In the case of period doubling, $2\beta^{(2)}\simeq 21$ for one-dimensional maps [1] and $2\beta^{(2)}\simeq 60$ for two-dimensional area-preserving maps. It is our purpose to find the scaling limits of the power spectra of higher period $p$-multiplings and see how they vary with $p$.

In the bifurcation of higher period multiplings, unlike period-doubling, the original stable mother orbit remains stable while one of the two daughter orbits is stable and the other one unstable [3,4]. Therefore, there exist two kinds of critical orbits at the period multipling accumulation point: one stable and the other unstable [5–8].We compute the power spectrum of these two kinds of critical orbits and the results for them are the same.

To check consistency, we have employed two methods to compute the power spectrum: the autocorrelation function method and the fast Fourier transform (FFT) method. The results of the two methods agree very well (see table 1). In our numerical calculation $2\beta^{(p)}$ was computed to order $n=10$ for $p=3$, $n=8$ for $p=4$ and 5, and $n=7$ for $p=6$.

It is seen that the scaling ratio $2\beta^{(p)}$ increases with $p$ in period multipling. However the rate of increase seems to slow down and approach a limiting value $\sim 1.15$. Of course, this observation is very tentative; it is nevertheless interesting.

The results reported in this work may be of relevance to future experiments involving Hamiltonian systems. However, the increasingly large values of $2\beta^{(p)}$ indeed make it quite difficult to observe those bifurcations in the asymptotic regime unless very precise measurements can be made.

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