Intermittent Transition to Chaos in Coupled Maps

Sang-Yoon KIM*

Department of Physics, Kangwon National University, Chunchon 200-701

(Received 23 July 1998)

We study the critical behavior associated with intermittency in two coupled one-dimensional maps by using a renormalization method. Two fixed points of a renormalization transformation are found. The relevant "coupling eigenvalue" associated with the coupling perturbation varies, depending on the kinds of fixed points, while the relevant eigenvalue associated with the scaling of the control parameter of the uncoupled one-dimensional map is common to all fixed points. These renormalization results are also confirmed for the dissipative-coupling case.

An intermittent transition to chaos in the one-dimensional (1D) map occurs in the vicinity of a saddle-node bifurcation [1]. Intermittency just preceding a saddle-node bifurcation to a periodic attractor is characterized by the occurrence of alternating bursts of regular and irregular behavior. Scaling relations for the average duration of regular behavior in the presence of noise were first established [2] by considering a Langevin equation describing the map near the intermittency threshold and by using Fokker-Plank techniques. The same scaling results were later found [3] by using the same renormalization-group equation [4] as used for period doubling, but with boundary conditions appropriate to a saddle-node bifurcation.

Recently, much effort has been made to generalize the scaling results of period doubling for the 1D map to the coupled 1D maps [5–8], which are used to simulate spatially extended systems with effectively many degrees of freedom [9]. It has been found that the critical scaling behaviors of period doubling in the coupled 1D maps are much richer than those in the uncoupled 1D map [8]. These results for the abstract system of the coupled 1D maps are also confirmed in the real system of coupled oscillators [10]. In a similar way, the scaling results of the higher period p-tuplings ($p = 3, 4, ...$) in the 1D map are also generalized to the coupled 1D maps by using a renormalization method [11].

In this paper we are interested in another route to chaos via intermittency in coupled 1D maps. We employ the same renormalization method [8] developed for period doubling in the coupled 1D maps and study the critical behavior for intermittency in two coupled 1D maps. We thus find two fixed points of the renormalization transformation. They have the relevant eigenvalue associated with scaling of the control parameter of the uncoupled 1D map as a common one. However, the relevant "coupling eigenvalue" (CE) associated with the coupling perturbation varies, depending on the kind of fixed points. These two fixed points are also found to be associated with the critical behavior near a critical line segment. The fixed point with no relevant CE governs the critical behavior, associated with a "1D-like" intermittent transition to chaos inside the critical line. On the other hand, the other fixed point with one relevant CE governs the critical behavior at both endpoints, where the 1D-like intermittent transition to chaos ends.

We consider a map T consisting of two symmetrically-coupled 1D maps,

$$T : \begin{cases} x_{t+1} = F(x_t, y_t) = f(x_t) + g(x_t, y_t), \\ y_{t+1} = F(y_t, x_t) = f(y_t) + g(y_t, x_t). \end{cases} \quad (1)$$

Here $f(x)$ is a 1D map with a control parameter ϵ and satisfies the boundary conditions [3],

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1, \quad (2)$$

which are appropriate to a synchronous saddle-node bifurcation at the origin for a threshold value of ϵ, and $g(x, y)$ is a coupling function obeying the condition

$$g(x, x) = 0 \qquad \text{for any } x. \quad (3)$$

We now employ the same renormalization transformation as in the period-doubling case [8], but with the changed boundary conditions (2). The renormalization transformation $\mathcal{N}$ for a coupled map T consists of the squaring (T^2) and rescaling (B) operators:

$$\mathcal{N}(T) \equiv BT^2B^{-1}. \quad (4)$$

Since we consider only synchronous orbits, the rescaling operator is of the form,

*E-mail: sykim@cc.kangwon.ac.kr

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad (5)$$

where α is a rescaling factor.

Applying the renormalization operator $\mathcal{N}$ to the coupled map (1) n times, we obtain an n-times renormalized map T_n of the form,

$$T_n : \begin{cases} x_{t+1} = F_n(x_t, y_t) = f_n(x_t) + g_n(x_t, y_t), \\ y_{t+1} = F_n(y_t, x_t) = f_n(y_t) + g_n(y_t, x_t). \end{cases} \qquad (6)$$

Here f_n and g_n are the uncoupled and coupling parts of the n-times renormalized function F_n, respectively. They satisfy the following recurrence equations [8]:

$$f_{n+1}(x) = \alpha f_n\left(f_n\left(\frac{x}{\alpha}\right)\right), \qquad (7)$$

$$\begin{aligned} g_{n+1}(x, y) = {} & \alpha f_n\left(f_n\left(\frac{x}{\alpha}\right) + g_n\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right)\right) \\ & + \alpha g_n\left(f_n\left(\frac{x}{\alpha}\right) + g_n\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), f_n\left(\frac{y}{\alpha}\right)\right. \\ & \left. + g_n\left(\frac{y}{\alpha}, \frac{x}{\alpha}\right)\right) - \alpha f_n\left(f_n\left(\frac{x}{\alpha}\right)\right). \end{aligned} \qquad (8)$$

Then Eqs. (7) and (8) define a renormalization operator $\mathcal{R}$ for transforming a pair of functions (f, g): $(f_{n+1}, g_{n+1}) = \mathcal{R}(f_n, g_n)$.

A critical map T_c with the nonlinearity and coupling parameters set to their critical values is attracted to a fixed map T^* under iterations of the renormalization transformation $\mathcal{N}$

$$T^* : \begin{cases} x_{t+1} = F^*(x_t, y_t) = f^*(x_t) + g^*(x_t, y_t), \\ y_{t+1} = F^*(y_t, x_t) = f^*(y_t) + g^*(y_t, x_t). \end{cases} \qquad (9)$$

Here (f^*, g^*) is a fixed point of the renormalization operator $\mathcal{R}$ and satisfies $(f^*, g^*) = \mathcal{R}(f^*, g^*)$. Note that the equation for f^* is just the fixed-point equation for intermittency with the boundary conditions (2) in the uncoupled 1D map. It was found in [3] that

$$f^*(x) = \frac{x}{1 - ax} = x + ax^2 + \cdots \qquad (10)$$

$$(a : \text{arbitrary constant})$$

is a fixed point of the transformation (7) with $\alpha = 2$. Here we consider only the most common case of quadratic tangency. Consequently, only the equation for the coupling fixed function g^* is left to be solved.

It is not easy to directly solve the equation for the coupling fixed function. We therefore introduce a tractable recurrence equation for a "reduced coupling function" defined by

$$G(x) \equiv \left. \frac{\partial g(x, y)}{\partial y} \right|_{y=x}. \qquad (11)$$

Differentiating the recurrence equation (8) for g with respect to y and setting $y = x$, we have

$$\begin{aligned} G_{n+1}(x) = {} & \left[f'_n\left(f_n\left(\frac{x}{\alpha}\right)\right) - 2G_n\left(f_n\left(\frac{x}{\alpha}\right)\right)\right] G_n\left(\frac{x}{\alpha}\right) \\ & + G_n\left(f_n\left(\frac{x}{\alpha}\right)\right) f'_n\left(\frac{x}{\alpha}\right). \end{aligned} \qquad (12)$$

Then Eqs. (7) and (12) define a "reduced renormalization operator" $\tilde{\mathcal{R}}$ for transforming a pair of functions (f, G) [8,11]: $(f_{n+1}, G_{n+1}) = \tilde{\mathcal{R}}(f_n, G_n)$. We look for a fixed point (f^*, G^*) of $\tilde{\mathcal{R}}$, which satisfies $(f^*, G^*) = \tilde{\mathcal{R}}(f^*, G^*)$. Here G^* is just the reduced coupling fixed function of g^* (*i.e.*, $G^*(x) = \partial g^*(x, y)/\partial y|_{y=x}$). Using a series-expansion method, we find two solutions for G^*:

$$G_I^*(x) = \frac{1}{2} + ax + \frac{3}{2}a^2x^2 + \cdots, \qquad (13)$$

$$G_{II}^*(x) = bx + \frac{b}{2}(5a - 2b)x^2 + \cdots, \qquad (14)$$

where a and b are arbitrary constants. Here we are able to sum the series of $G_I^*(x)$ and obtain the closed-form solution,

$$G_I^*(x) = \frac{1}{2} f^{*\prime}(x). \qquad (15)$$

However, unfortunately we cannot sum the series of $G_{II}^*(x)$, except for the cases $b = 0$ and $b = a$. In those cases we obtain the closed-form solutions,

$$G_{II}^*(x) = \begin{cases} 0 & \text{for } b = 0, \\ \frac{1}{2}[f^{*\prime}(x) - 1] & \text{for } b = a. \end{cases} \qquad (16)$$

Consider an infinitesimal perturbations (h, Φ) to a fixed point (f^*, G^*) of the reduced renormalization operator $\tilde{\mathcal{R}}$. Linearizing $\tilde{\mathcal{R}}$ at the fixed point, we obtain the recurrence equation for the evolution of (h, Φ): $(h_{n+1}, \Phi_{n+1}) = \tilde{\mathcal{L}}(h_n, \Phi_n)$. A pair of perturbation (h^*, Φ^*) is called an eigenperturbation with eigenvalue λ if $\tilde{\mathcal{L}}(h^*, \Phi^*) = \lambda(h^*, \Phi^*)$. All the fixed points (f^*, G^*) have a common relevant eigenvalue δ_1 $(= 4)$ associated with the scaling of the control parameter of the uncoupled 1D map. However, the relevant CE associated with the coupling perturbation depends on the kind of the fixed points as follows . For the case of the first fixed point (f^*, G_I^*), it has no relevant CE's, while for the case of the second fixed point (f^*, G_{II}^*), it has one relevant CE δ_2 $(= 2)$. These two fixed points are also found to be associated with the critical behavior near a critical line, as will be seen below. The first fixed point with no relevant CE's governs the critical behavior inside the critical line, whereas the second fixed point with one relevant CE governs the critical behavior at both endpoints.

To confirm the above renormalization results, we consider two dissipatively-coupled 1D maps M,

$$M : \begin{cases} X_{t+1} = W(X_t, Y_t) = u(X_t) + v(X_t, Y_t), \\ Y_{t+1} = W(Y_t, X_t) = u(Y_t) + v(Y_t, X_t), \end{cases} \qquad (17)$$

where the uncoupled 1D map u and the coupling function v are given by $u(X) = 1 - AX^2$ and $v(X, Y) = \frac{c}{2}[u(Y) - u(X)]$ (c is a coupling parameter), respectively. As an example, we consider the saddle-node bifurcation to synchronous orbits with period $p = 3$ occurring for $A = A_c = 1.75$. To study the intermittency associated with this bifurcation, we first consider the third iterate $M^{(3)}$ of M and then shift the orgin of coordinates (X, Y)

to one of the three synchronous fixed points (X^*, Y^*) for $A = A_c$ [$Y^* = X^* = u^{(3)}(X^*)$; $u^{(3)}$ is the third iterate of u]. Thus we obtain a map T of the form (1), where the uncoupled and coupling parts f and g are given by

$$f(x) = u^{(3)}(x + X^*) - X^*, \quad (18)$$

$$g(x,y) = W^{(3)}(x + X^*, y + Y^*) - u^{(3)}(x + X^*). \quad (19)$$

Near the region of the synchronous saddle-node bifurcation, $f(x)$ can be expanded about $x = 0$ and $A = A_c$:

$$f(x) \approx x + ax^2 + \epsilon, \quad (20)$$

where $a = \frac{1}{2}\partial^2 f / \partial x^2|_{x=0, A=A_c}$ and $\epsilon = \partial f/\partial A|_{x=0, A=A_c}(A - A_c)$. Hence this corresponds to the most common case of the quadratic tangency. The reduced coupling function $G(x)$ of $g(x,y)$ [defined in Eq. (11)] is also given by

$$G(x) = \frac{e}{2} f'(x), \quad e = c^3 - 3c^2 + 3c. \quad (21)$$

Consider a pair of initial functions (f_c, G) on the synchronous saddle-node bifurcation line $A = A_c$, where $f_c(x)$ is just the 1D critical map and $G(x) = \frac{e}{2} f_c'(x)$. By successive actions of the reduced renormalization transformation $\tilde{\mathcal{R}}$ on (f_c, G), we obtain

$$f_n(x) = \alpha f_{n-1}\Big(f_{n-1}\Big(\frac{x}{\alpha}\Big)\Big), \quad G_n(x) = \frac{e_n}{2} f_n'(x), \quad (22)$$

$$e_n = 2e_{n-1} - e_{n-1}^2, \quad (23)$$

where the rescaling factor is $\alpha = 2$, $f_0(x) = f_c(x)$, $G_0(x) = G(x)$, and $e_0 = e$. Here f_n converge to the 1D fixed function $f^*(x)$ of Eq. (11) as $n \to \infty$, because the nonlinearity parameter A is set to its critical value A_c.

The recurrence equation (23) for e has two fixed points e^*:

$$e^* = 0, 1. \quad (24)$$

Stability of a fixed point e^* is determined by its stability multiplier μ given by $\mu = de_n/de_{n-1}|_{e^*}$. The fixed point at $e^* = 1$ is superstable ($\mu = 0$), while the other one at $e^* = 0$ is unstable ($\mu = 2$). The basin of attraction to the superstable fixed point $e^* = 1$ is the open interval $(0, 2)$. That is, any initial e inside the interval $0 < e < 2$ converges to $e^* = 1$ under successive iterations of the transformation (23). The left end of the interval is just the unstable fixed point $e^* = 0$, which is also the image of the right endpoint under the recurrence equation (23). All the other points outside the interval diverge to minus infinity under iterations of the transformation (23).

It follows from the relation $e = e(c)$ in Eq. (21) that there exists a critical line segment joining the two end points $c_l = 0$ and $c_r = 2$ on the synchronous saddle-node bifurcation line $A = A_c$ in the $c - A$ plane. Inside the critical line segment, any pair of critical functions (f_c, G_c) is attracted to the first fixed point (f^*, G_I^*) (corresponding to the case $e^* = 1$) under iterations of $\tilde{\mathcal{R}}$.

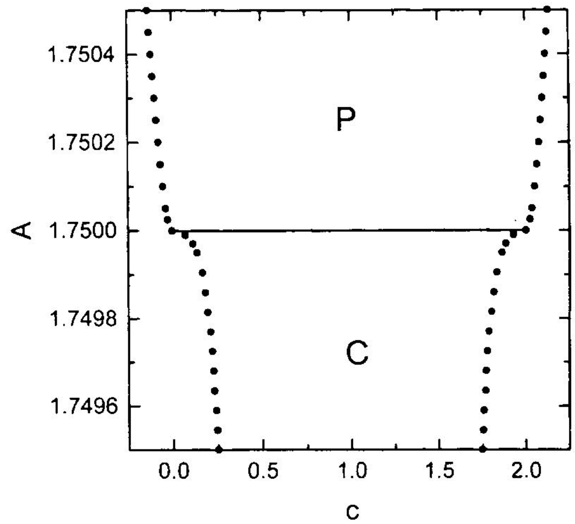

Fig. 1. Phase diagram for a dissipative-coupling case. Here the solid circles denote the data points on the $\sigma_\perp = 0$ curve. The region enclosed by the $\sigma_\perp = 0$ curve is divided into two parts denoted by P and C. A synchronous period-3 (chaotic) attractor with $\sigma_\parallel < 0$ ($\sigma_\parallel > 0$) exists in the part P (C). The boundary curve denoted by the solid line between the P and C parts is just the critical line segment.

This first fixed point has no relevant CE's, because the fixed point $e^* = 1$ is superstable. On the other hand, the pair of critical functions (f_c, G_c) at each end point of the critical line converges to the second fixed point (f^*, G_{II}^*) (corresponding to the case $e^* = 0$) under iterations of $\tilde{\mathcal{R}}$. This second fixed point has one relevant CE δ_2 (= 2), because the fixed point $e^* = 0$ is an unstable one with the stability multiplier $\mu = 2$.

Figure 1 shows a phase diagram near the critical line denoted by the solid line. The diagram is obtained by calculating the Lyapunov exponents. For the case of a synchronous orbit, its two Lyapunov exponents are given by

$$\sigma_\parallel(A) = \lim_{m\to\infty} \frac{1}{m}\sum_{t=0}^{m-1} \ln|u'(X_t)|, \quad (25)$$

$$\sigma_\perp(A, c) = \lim_{m\to\infty} \frac{1}{m}\sum_{t=0}^{m-1} \ln|(1-c)u'(X_t)|. \quad (26)$$

Here $\sigma_\parallel(\sigma_\perp)$ denotes the mean exponential rate of divergence of nearby orbits along (across) the synchronization line $Y = X$. Hereafter, $\sigma_\parallel$ ($\sigma_\perp$) will be referred to as tangential (transversal) Lyapunov exponents. Note also that $\sigma_\parallel$ is just the Lyapunov exponent for the 1D case, and the coupling affects only $\sigma_\perp$.

The data points on the $\sigma_\perp = 0$ curve are denoted by solid circles in Fig. 1. A synchronous orbit on the synchronization line becomes a synchronous attractor with $\sigma_\perp < 0$ inside the $\sigma_\perp = 0$ curve. The type of this synchronous attractor is determined according to the sign of $\sigma_{\|}$. A synchronous period-3 orbit with $\sigma_{\|} < 0$ becomes a synchronous periodic attractor above the critical line segment, while a synchronous chaotic attractor with $\sigma_{\|} > 0$ exists below the critical line segment. The periodic and chaotic parts in the phase diagram are denoted by P and C, respectively. There also exists a synchronous period-3 attractor with $\sigma_{\|} = 0$ on the critical line segment between the two parts.

We first consider a transition to chaos occurring when crossing the critical line at any interior point with $c_l < c < c_r$. We fix the value of c at some interior point and vary the control parameter ϵ_A ($\equiv A_c - A$) of the uncoupled 1D map. For $\epsilon_A < 0$ there exists a synchronous period-3 attractor on the invariant synchronization line $Y = X$. However, as ϵ_A is increased from zero, the periodic attractor disappears and a new chaotic attractor appears on the $Y = X$ line. The motion on this synchronous chaotic attractor is characterized by the occurrence of random alternations of laminar and turbulent behaviors on the $Y = X$ line. This is just the intermittency occurring in the uncoupled 1D map, because the motion on the $Y = X$ line is the same as that for the uncoupled 1D case. Thus, a "1D-like" intermittent transition to chaos occurs near interior points of the critical line segment. The scaling relations of the mean duration of the laminar phase $\bar{l}$ and the tangential Lyapunov exponent $\sigma_{\|}$ for the synchronous chaotic attractor are obtained from the common relevant eigenvalue δ_1 $(= 4)$ of the first fixed point (f^*, G_I^*) with no relevant CE's, as in the 1D case [3]:

$$\bar{l}(\epsilon_A) \sim \epsilon_A^{-\nu}, \ \ \sigma_{\|}(\epsilon_A) \sim \epsilon_A^{\nu}; \ \ \nu = \log 2/\log \delta_1 = 0.5. \quad (27)$$

The 1D-like intermittent transition to chaos ends at both ends c_l and c_r of the critical line segment. We fix the value of the control parameter $A = A_c (= 1.75)$ and study the critical behaviors near the two endpoints by varying the coupling parameter c. Inside the critical line segment $(c_l < c < c_r)$, a synchronous period-3 attractor exists on the synchronization line $Y = X$. However, as the coupling parameter c passes through c_l or c_r, the transversal Lyapunov exponent $\sigma_{\perp}$ of the synchronous periodic orbit increases from zero, and hence the coupling leads to desynchronization of the interacting systems. Thus the synchronous period-3 orbit ceases to be an attractor outside the critical line segment, and a new asynchronous (out-of-phase) attractor appears. The scaling relation of the transversal Lyapunov exponent $\sigma_{\perp}$ near both ends c_l and c_r is obtained from the relevant CE δ_2 $(= 2)$ of the second fixed point (f^*, G_{II}^*):

$$\sigma_{\perp}(\epsilon_A) \sim \epsilon_c^{\nu}; \ \ \nu = \log 2/\log \delta_2 = 1, \quad (28)$$

where $\epsilon_c = c_l - c$ or $c - c_r$.

To sum up, we have studied the intermittent transition to chaos in two coupled 1D maps by using a renormalization method. It has been found that there exist two fixed points of the renormalization transformation. The fixed point with no relevant CE's governs the critical behaviors, associated with the 1D-like intermittent transition to chaos inside a critical line. On the other hand, the other fixed point with one relevant CE governs the critical behaviors at both ends of the critical line, where the 1D-like intermittent transition to chaos ends. It is also expected that the results in the abstract system of the coupled 1D maps may be confirmed in the real system of coupled oscillators, as in the period-doubling case [10]. An extended version of this work, including a detailed account of the renormalization results and the critical scaling behaviors near the critical line, a generalization to the nonquadratic-tangency cases, an extension to many coupled maps, and so on, will be given elsewhere [12].

ACKNOWLEDGMENTS

This work was supported by the Korea Research Foundation under Project No. 1997-001-D00099.

REFERENCES

[1] P. Manneville and Y. Pomeau, Phys. Lett. **A75**, 1 (1979); Physica **D1**, 219 (1980); Y. Pomeau and P. Manneville, Commun. Math. Phys. **74**, 189 (1980).

[2] J.-P. Eckmann, L. Thomas and P. Wittwer, J. Phys. **A14**, 3153 (1981); J. E. Hirsh, B. A. Hubermann and D. J. Scalapino, Phys. Rev. **A25**, 519 (1982).

[3] J. E. Hirsh, M. Nauenberg and D. J. Scalapino, Phys. Lett. **A87**, 391 (1982); B. Hu and J. Rudnick, Phys. Rev. Lett. **48**, 1645 (1982).

[4] M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979).

[5] I. Waller and R. Kapral, Phys. Rev. **A30**, 2047 (1984); R. Kapral, Phys. Rev. **A31**, 3868 (1985).

[6] S. P. Kuznetsov, Radiophys. Quantum Electron. **28**, 681 (1985); S. P. Kuznetsov and A. S. Pikovsky, Physica **D19**, 384 (1986); H. Kook, F. H. Ling and G. Schmidt, Phys. Rev. **A43**, 2700 (1991).

[7] I. S. Aranson, A. V. Gaponov-Grekhov and M. I. Rabinovich, Physica **D33**, 1 (1988).

[8] S.-Y. Kim and H. Kook, Phys. Rev. **A46**, R4467 (1992); Phys. Lett. **A178**, 258 (1993); Phys. Rev. **E48**, 785 (1993); S.-Y. Kim, Phys. Rev. **E49**, 1745 (1994).

[9] K. Kaneko, in *Theory and Applications of Coupled Map Lattices*, edited by K. Kaneko (John Wiley & Sons, New York, 1992), p. 1, and references cited therein.

[10] S.-Y. Kim and K. Lee, Phys. Rev. **E54**, 1237 (1996).

[11] S.-Y. Kim, Phys. Rev. **E52**, 1206 (1995); J. Korean Phys. Soc. **28**, 228 (1995); J. Korean Phys. Soc. **28**, 534 (1995); Phys. Rev. **E54**, 3393 (1996).

[12] S.-Y. Kim (accepted for publication in Phys. Rev. E).