Universal Behavior of Period p-Tuplings in Coupled Maps

Sang-Yoon KIM*

Department of Physics, Kangwon National University, Chunchon 200-701

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We study the universal scaling behavior of all period p-tuplings ($p = 2, 3, 4, \ldots$) in two coupled one-dimensional maps with a single maximum of order z. The effect of the maximum-order z on the critical behavior associated with coupling is investigated by a renormalization method. We find three (five) fixed points of the renormalization method for even (odd) p. However, relevant "coupling eigenvalues" associated with coupling perturbations vary depending on the order z only for one (two) fixed point(s) in the case of even (odd) p, whereas they are independent of z for the other two (three) fixed points. These renormalization results are also confirmed by a direct numerical method.

Universal scaling behaviors of period p-tupling ($p = 2, 3, 4, \ldots$) sequences of p^n-cycles (*i.e.*, orbits of period p^n) ($n = 1, 2, \ldots$) have been found in a one-parameter family of one-dimensional (1D) unimodal maps with a single maximum of order z ($z > 1$) [1–10]. As an example, consider a 1D map with a maximum of order z at $x = 0$,

$$x_{t+1} = f(x_t) = 1 - A\,|x_t|^z, \quad z > 1, \tag{1}$$

where x_t denotes a state variable at a discrete time t. For all $z > 1$, an infinite sequence of period doublings ($p = 2$) accumulates at a finite parameter value A_∞ and exhibits an asymptotic scaling behavior.

The parameter interval between A_∞ and the final boundary-crisis point ($A = 2$) beyond which no periodic or chaotic attractors can be found within the unimodality interval is called the "chaotic" region. Besides the period-doubling sequence, there exist infinitely many higher period p-tupling ($p = 3, 4, 5, \ldots$) sequences inside the chaotic region. These higher period p-tupling sequences also exhibit their own asymptotic scaling behaviors near their accumulation points $A_\infty^{(p)}$. However, the critical behaviors characterized by the parameter and the orbital scaling factors, δ and α, vary depending on p. Moreover, for each period p-tupling case, the maximum-order z affects the critical behavior; consequently, the values of $A_\infty^{(p)}$, δ, and α vary depending on z [1–10]. Thus, the order z determines universality classes in each period p-tupling case.

In this paper, we are interested in the critical behaviors of all period p-tuplings ($p = 2, 3, 4, \ldots$) in two symmetrically coupled 1D maps. The coupled maps are used as models of coupled nonlinear oscillators such as Josephson-junction arrays, chemically reacting cells, and so on [11]. The critical behavior of period doublings ($p = 2$) in such coupled 1D maps was first studied for the quadratic-maximum case ($z = 2$) [12,13]; then, the results for the $z = 2$ case were extended to all even-order cases ($z = 2, 4, 6, \ldots$) [14]. However, the critical behaviors of all the other higher period p-tuplings ($p = 3, 4, 5, \ldots$) in the coupled 1D maps were studied only for the quadratic-maximum case [15,16]. Here, we extend the results of the higher period p-tuplings for the $z = 2$ case to all even-order cases by a renormalization method. The renormalization results are also confirmed by a direct numerical method.

Consider a map T consisting of two identical 1D maps coupled symmetrically:

$$T: \begin{cases} x_{t+1} = F(x_t, y_t) = f(x_t) + g(x_t, y_t) \\ y_{t+1} = F(y_t, x_t) = f(y_t) + g(y_t, x_t) \end{cases} \tag{2}$$

where $f(x)$ is the 1D map (1), and $g(x, y)$ is a coupling function. The uncoupled 1D map f satisfies the normalization condition, $f(0) = 1$, and the coupling function g obeys the condition $g(x, x) = 0$ for any x. Here, we consider only the analytic cases, *i.e.*, the cases of even-order z ($z = 2, 4, 6, \ldots$).

The two-coupled map (2) is invariant under the exchange of coordinates such that $x \leftrightarrow y$. The set of all points which are invariant under the exchange of coordinates forms a symmetry line $y = x$. An orbit is called an "in-phase" orbit if it lies on the symmetry line, *i.e.*, it satisfies

$$x_t = y_t \quad \text{for all } t. \tag{3}$$

Otherwise, it is called an "out-of-phase" orbit. Here, we study only in-phase orbits.

Stability of an in-phase orbit with period q is determined from the Jacobian matrix J of T^q, which is the

*Electronic address: sykim@cc.kangwon.ac.kr

q-product of the Jacobian matrix DT of T along the orbit:

$$\begin{aligned} J &= \prod_{t=1}^{q} DT(x_t, x_t) \\ &= \prod_{t=1}^{q} \begin{pmatrix} f'(x_t) - G(x_t) & G(x_t) \\ G(x_t) & f'(x_t) - G(x_t) \end{pmatrix} \end{aligned} \quad (4)$$

where the prime denotes a derivative, and $G(x) = \partial g(x,y)/\partial y \mid_{y=x}$; hereafter, $G(x)$ will be referred to as the "reduced coupling function" of $g(x,y)$. The eigenvalues of J, called the stability multipliers of the orbit, are

$$\lambda_1 = \prod_{t=1}^{q} f'(x_t), \quad \lambda_2 = \prod_{t=1}^{q} [f'(x_t) - 2G(x_t)]. \quad (5)$$

Note that the first stability multiplier λ_1 is just that of the uncoupled 1D map and the coupling affects only the second stability multiplier λ_2, which may be called the "coupling stability multiplier." An in-phase orbit is stable only when the moduli of both multipliers are less than or equal to unity, *i.e.*, $-1 \leq \lambda_i \leq 1$ for $i = 1, 2$.

We now consider the period p-tupling renormalization transformation $\mathcal{N}$, which is composed of the p-times iterating $(T^{(p)})$ and rescaling (B) operators:

$$\mathcal{N}(T) \equiv BT^{(p)}B^{-1}. \quad (6)$$

Here, the rescaling operator B is

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad (7)$$

because we consider only in-phase orbits ($x_t = y_t$ for all t).

Applying the renormalization operator $\mathcal{N}$ to the coupled map (2) n times, we obtain the n-times renormalized map T_n of the form

$$T_n : \begin{cases} x_{t+1} = F_n(x_t, y_t) = f_n(x_t) + g_n(x_t, y_t) \\ y_{t+1} = F_n(y_t, x_t) = f_n(y_t) + g_n(y_t, x_t). \end{cases} \quad (8)$$

Here, f_n and g_n are the uncoupled and coupling parts of the n-times renormalized function F_n, respectively. They satisfy the following recurrence equations:

$$f_{n+1}(x) = \alpha f_n^{(p)}\left(\frac{x}{\alpha}\right), \quad (9)$$

$$g_{n+1}(x,y) = \alpha F_n^{(p)}\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) - \alpha f_n^{(p)}\left(\frac{x}{\alpha}\right), \quad (10)$$

where

$$f_n^{(p)}(x) = f_n(f_n^{(p-1)}(x)), \quad (11)$$

$$F_n^{(p)}(x,y) = F_n(F_n^{(p-1)}(x,y), F_n^{(p-1)}(y,x)), \quad (12)$$

and the rescaling factor is chosen to preserve the normalization condition $f_{n+1}(0) = 1$, *i.e.*, $\alpha = 1/f_n^{(p-1)}(1)$. The recurrence relations (9) and (10) define a renormalization operator $\mathcal{R}$ for transforming a pair of functions (f, g):

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = \mathcal{R} \begin{pmatrix} f_n \\ g_n \end{pmatrix}. \quad (13)$$

A critical map T_c with the nonlinearity and coupling parameters set to their critical values is attracted to a fixed map T^* under iterations of the renormalization transformation $\mathcal{N}$:

$$T^* : \begin{cases} x_{t+1} = F^*(x_t, y_t) = f^*(x_t) + g^*(x_t, y_t) \\ y_{t+1} = F^*(y_t, x_t) = f^*(y_t) + g^*(y_t, x_t). \end{cases} \quad (14)$$

Here, (f^*, g^*) is a fixed point of the renormalization operator $\mathcal{R}$ with $\alpha = 1/f^{*(p-1)}(1)$, which satisfies $(f^*, g^*) = \mathcal{R}(f^*, g^*)$. Note that $f^*(x)$ is just the fixed function in the 1D map case, which varies depending on p [4,5,7,8]. Only the equation for the coupling fixed function $g^*(x,y)$ is, therefore, left to be solved. One trivial solution is $g^*(x,y) = 0$. In this zero-coupling case, the fixed map (14), which is associated with the critical behavior at the zero-coupling critical point, consists of two uncoupled 1D fixed maps.

However, it is not easy to directly find coupling fixed functions other than the zero-coupling fixed function $g^*(x,y) = 0$. We, therefore, introduce a tractable recurrence equation for a reduced coupling function $G(x) = \partial g(x,y)/\partial y|_{y=x}$. Differentiating the recurrence equation (10) for $g(x,y)$ with respect to y and setting $y = x$, we obtain a recurrence equation for $G(x)$:

$$\begin{aligned} G_{n+1}(x) &= F_{n,2}^{(p)}\left(\frac{x}{\alpha}\right) \\ &= F_{n,2}^{(p-1)}\left(\frac{x}{\alpha}\right)\left[f_n'\left(f_n^{(p-1)}\left(\frac{x}{\alpha}\right)\right) - 2G_n\left(f_n^{(p-1)}\left(\frac{x}{\alpha}\right)\right)\right] \\ &\quad + f_n^{(p-1)'}\left(\frac{x}{\alpha}\right) G_n\left(f_n^{(p-1)}\left(\frac{x}{\alpha}\right)\right) \end{aligned} \quad (15)$$

where the subscript 2 of F_n denotes a partial derivative with respect to the second argument. Then, Eqs. (9) and (15) define a "reduced renormalization operator" $\tilde{\mathcal{R}}$ for transforming a pair of functions (f, G):

$$\begin{pmatrix} f_{n+1} \\ G_{n+1} \end{pmatrix} = \tilde{\mathcal{R}} \begin{pmatrix} f_n \\ G_n \end{pmatrix}. \quad (16)$$

We look for fixed points (f^*, G^*) of $\tilde{\mathcal{R}}$ which satisfy $(f^*, G^*) = \tilde{\mathcal{R}}(f^*, G^*)$. Here, f^* is just the 1D fixed function, and G^* is the reduced coupling fixed function of g^*; *i.e.*, $G^*(x) = \partial g^*(x,y)/\partial y \mid_{y=x}$. As in the quadratic maximum case ($z = 2$) [16], We find three (five) solutions for G^* in the case of even (odd) p:

$$G^*(x) = 0, \quad (17)$$

$$G^*(x) = \frac{1}{2} f^{*\prime}(x), \quad (18)$$

$$G^*(x) = \frac{1}{2}[f^{*\prime}(x) - 1], \quad (19)$$

$$G^*(x) = \frac{1}{2}[f^{*\prime}(x) + 1], \quad (20)$$

$$G^*(x) = f^{*\prime}(x) \quad (21)$$

where the solutions (17)-(19) exist for any p, but the solutions (20)-(21) exist only for odd p. The first solution (17) corresponds to the zero-coupling case, while the other solutions (18)-(21) are associated with critical behaviors at critical points other than the zero-coupling critical point [16].

Consider an infinitesimal, reduced coupling perturbation $(0, \Phi(x))$ to a fixed point (f^*, G^*) of $\tilde{\mathcal{R}}$. When $f_n(x) = f^*(x)$ and $G_n(x) = G^*(x)$, the function $F_{n,2}^{(p)}(\frac{x}{\alpha})$ of Eq. (15) will be denoted by $F_{*,2}^{(p)}(\frac{x}{\alpha})$. We then examine the evolution of a pair of functions, $(f^*(x), G^*(x)+\Phi(x))$ under the reduced renormalization transformation $\tilde{\mathcal{R}}$. In the linear approximation we obtain a reduced linearized operator $\tilde{\mathcal{L}}$ for transforming the reduced coupling perturbation Φ:

$$\begin{aligned}\Phi_{n+1}(x) &= [\tilde{\mathcal{L}}\Phi_n](x) \\ &= \delta F_{2,n}\left(\frac{x}{\alpha}\right) = \left[F_{n,2}^{(p)}\left(\frac{x}{\alpha}\right) - F_{*,2}^{(p)}\left(\frac{x}{\alpha}\right)\right]\Big|_{\rm linear} \\ &= \left[f^{*'}\left(f^{*(p-1)}\left(\frac{x}{\alpha}\right)\right) - 2G^*\left(f^{*(p-1)}\left(\frac{x}{\alpha}\right)\right)\right] \\ &\quad \times \delta F_{n,2}^{(p-1)}\left(\frac{x}{\alpha}\right) + \left[f^{*(p-1)'}\left(\frac{x}{\alpha}\right) - 2F_{*,2}^{(p-1)}\left(\frac{x}{\alpha}\right)\right] \\ &\quad \times \Phi_n\left(f^{*(p-1)}\left(\frac{x}{\alpha}\right)\right). \end{aligned} \tag{22}$$

Here the variation $\delta F_{n,2}^{(p)}(\frac{x}{\alpha})$ is introduced as a linear term (denoted by $[F_{n,2}^{(p)}(\frac{x}{\alpha}) - F_{*,2}^{(p)}(\frac{x}{\alpha})]_{\rm linear}$) in Φ for the deviation of $F_{n,2}^{(p)}(\frac{x}{\alpha})$ from $F_{*,2}^{(p)}(\frac{x}{\alpha})$. If the reduced coupling perturbation $\Phi^*(x)$ satisfies

$$\nu\Phi^*(x) = [\tilde{\mathcal{L}}\Phi^*](x), \tag{23}$$

then it is called a reduced coupling eigenperturbation with a coupling eigenvalue (CE) ν.

We first show that the CE's are independent of the order z for the second, third, and fourth solutions of $G^*(x)$ [see Eqs. (18)-(20)]. In the case of the second solution $G^*(x) = \frac{1}{2}f^{*'}(x)$, the reduced linearized operator $\tilde{\mathcal{L}}$ becomes a null operator independently of z, because the right-hand side of Eq. (23) becomes zero. Therefore, no relevant CE's exist. For the third case $G^*(x) = \frac{1}{2}[f^{*'}(x) - 1]$, the CE equation (23) becomes

$$\begin{aligned}\nu\Phi^*(x) &= [\tilde{\mathcal{L}}\Phi^*](x) = \delta F_2^{(p)}\left(\frac{x}{\alpha}\right) \\ &= \sum_{i=0}^{p-1}\Phi^*\left(f^{*(i)}\left(\frac{x}{\alpha}\right)\right). \end{aligned} \tag{24}$$

When $\Phi^*(x)$ is a nonzero constant function, *i.e.*, $\Phi^*(x) = b$ (b : nonzero constant), a relevant CE, $\nu = p$, exists independently of z. For the fourth case $G^*(x) = \frac{1}{2}[f^{*'}(x)+1]$, which exists only for the case of odd p, the CE equation (23) is just that of Eq. (24). Therefore, it has the same CE, $\nu = p$, as that for the case $G^*(x) = \frac{1}{2}[f^{*'}(x) - 1]$, independently of z.

The remaining solutions are the first and fifth ones; *i.e.*, $G^*(x) = 0$ and $G^*(x) = f^{*'}(x)$. As mentioned earlier, the solution $G^*(x) = 0$ is associated with the critical behavior at the zero-coupling critical point, while the solution $G^*(x) = f^{*'}(x)$, which exists only for odd p, is associated with the critical behavior at other (nonzero-coupling) critical points. In both the cases $G^*(x) = 0$ and $f^{*'}(x)$, we have the same CE equation, composed of p terms:

$$\begin{aligned}\nu\Phi^*(x) &= [\tilde{\mathcal{L}}\Phi^*](x) = \delta F_2^{(p)}\left(\frac{x}{\alpha}\right) \\ &= \sum_{i=0}^{p-1} f^{*(i)'}\left(\frac{x}{\alpha}\right)\Phi^*\left(f^{*(i)}\left(\frac{x}{\alpha}\right)\right) \\ &\quad \times f^{*(p-1-i)'}\left(f^{*(i+1)}\left(\frac{x}{\alpha}\right)\right) \end{aligned} \tag{25}$$

where $f^{(0)}(x) = x$. Relevant CE's of Eq. (25) vary depending on the order z, as will be seen below. Thus, relevant CE's vary depending on the order z only for one (two) fixed point(s) in the case of even (odd) p.

An eigenfunction $\Phi^*(x)$ can be separated into two components, $\Phi^*(x) = \Phi^{*(1)}(x) + \Phi^{*(2)}(x)$ with $\Phi^{*(1)}(x) \equiv a_0^* + a_1^*x + \cdots + a_{z-2}^*x^{z-2}$ and $\Phi^{*(2)}(x) \equiv a_{z-1}^*x^{z-1} + a_z^*x^z + \cdots$, and the 1D fixed function f^* is a polynomial in x^z; *i.e.*, $f^*(x) = 1 + c_z^*x^z + c_{2z}^*x^{2z} + \cdots$. Substituting the functions Φ^*, f^*, and $f^{*'}$ into the CE equation (25), the structure is

$$\nu a_k^* = \sum_l M_{kl}(\{c^*\})a_l^*,\ k, l = 0, 1, 2, \ldots . \tag{26}$$

Each a_l^* ($l = 0, 1, 2, \ldots$) in the first term (the $i = 0$ case) on the right-hand side of Eq. (25) is involved only in the determination of coefficients of monomials x^k with $k = l + mz$ ($m = 0, 1, 2, \ldots$), while each a_l^* in all the remaining $(p-1)$ terms (the cases of $i = 1, \ldots, p-1$) is invloved only in the determination of the coefficients of the monomials x^k with $k = (z-1) + mz$. Therefore, any a_l^* with $l \geq z-1$ (on the right-hand side) cannot be involved in the determination of the coefficients of the monomials x^k with $k < z-1$, which implies that Eq. (26) is of the form

$$\nu\begin{pmatrix}\Phi^{*(1)} \\ \Phi^{*(2)}\end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ M_3 & M_2\end{pmatrix}\begin{pmatrix}\Phi^{*(1)} \\ \Phi^{*(2)}\end{pmatrix} \tag{27}$$

where M_1 is a $(z-1) \times (z-1)$ matrix, $\Phi^{*(1)} \equiv (a_0^*, \ldots, a_{z-2}^*)$, and $\Phi^{*(2)} \equiv (a_{z-1}^*, a_z^*, \ldots)$. From the reducibility of the matrix M into a semi-block form, it follows that to determine the eigenvalues of M it is sufficient to solve the eigenvalue problems for the two submatrices M_1 and M_2 independently.

We first solve the eigenvalue equation of M_1 ($\nu\Phi^{*(1)} = M_1\Phi^{*(1)}$); *i.e.*,

$$\nu a_k^* = \sum_l M_{kl}(\{c^*\})a_l^*,\ \ k, l = 0, \ldots, z-2. \tag{28}$$

Note that this submatrix M_1 is diagonal. Hence, its eigenvalues are just the diagonal elements:

$$\nu_k = M_{kk} = \frac{\prod_{l=1}^{p-1} f^{*'}(f^{*(l)}(0))}{\alpha^k} = \alpha^{z-1-k}, \quad k=0,\ldots,z-2. \quad (29)$$

Notice that all ν_k's are relevant eigenvalues.

Although ν_k is also an eigenvalue of M, $(\Phi_k^{*(1)}, 0)$ cannot be an eigenvector of M, because a third submatrix exists M_3 in M [see Eq. (27)]. Therefore, an eigenfunction $\Phi_k^*(x)$ in Eq. (25) with eigenvalue ν_k is a polynomial with a leading monomial of degree k; *i.e.*, $\Phi_k^*(x) = \Phi_k^{*(1)}(x)+\Phi_k^{*(2)}(x) = a_k^* x^k + a_{z-1}^* x^{z-1} + a_z^* x^z + \cdots$, where $a_k^* \neq 0$.

We next solve the eigenvalue equation of M_2 ($\nu\Phi^{*(2)} = M_2\Phi^{*(2)}$); *i.e.*,

$$\nu a_k^* = \sum_l M_{kl}(\{c^*\}) a_l^*, \quad k,l = z-1, z, \ldots . \quad (30)$$

Unlike the case of M_1, $(0, \Phi^{*(2)})$ can be an eigenvector of M with eigenvalue ν. Then, its corresponding function $\Phi^{*(2)}(x)$ is an eigenperturbation with eigenvalue ν which satisfies Eq. (25). One can easily see that $\Phi^{*(2)}(x) = f^{*\prime}(x)$ is an eigenfunction with CE, $\nu = p$, which is the zth relevant CE in addition to those in Eq. (29). It is also found that an infinite number of additional (coordinate change) eigenfunctions $\Phi^{*(2)}(x) = f^{*\prime}(x)[f^{*n}(x) - x^n]$ exist with irrelevant CE's α^{-n} $(n = 1, 2, \ldots)$, which are associated with coordinate changes. We conjecture that together with the z relevant (noncoordinate change) CE's, these irrelevant CE's give the whole spectrum of the reduced linearized operator $\tilde{\mathcal{L}}$ of Eq. (22) and that the spectrum is complete.

We now examine the effect of the CE's on the coupling stability multipliers for two kinds of couplings. We first consider the two coupled 1D maps (2) with $f(x) = f_c(x)$ and $g(x,y) = \varepsilon\varphi(x,y)$. Here, $f_c(x)$ is the 1D critical map with the nonlinearity parameter set to its critical value $A = A_\infty^{(p)}$, and ε is an infinitesimal coupling parameter. The map for $\varepsilon = 0$ is just the critical map T_c at the zero-coupling critical point consisting of two uncoupled 1D critical maps f_c. It is attracted to the zero-coupling fixed map (14) with $F^*(x,y) = f^*(x)$ under iterations of the renormalization transformation $\mathcal{N}$ of Eq. (6). Hence, the reduced coupling function $G(x)$ [$= \varepsilon\Phi \equiv \partial\varphi(x,y)/\partial y|_{y=x}$] corresponds to an infinitesimal reduced coupling perturbation to the reduced coupling fixed function $G^*(x) = 0$.

We next consider the two coupled 1D maps (2) with $f(x) = f_c(x)$ and $g(x,y) = f_c(y) - f_c(x) + \varepsilon\varphi(x,y)$ for odd p. The critical map T_c for $\varepsilon = 0$ is attracted to the fixed map (14) with $F^*(x,y) = f^*(y)$ under iterations of $\mathcal{N}$. Note that the coupling fixed function for this case is given by $g^*(x,y) = f^*(y) - f^*(x)$. Since the reduced coupling function $G(x)$ [$= f_c'(x)$] for $\varepsilon = 0$ converges to $G^*(x) = f^{*\prime}(x)$ under iterations of $\tilde{\mathcal{R}}$, $\varepsilon\Phi(x)$ can be regarded as an infinitesimal, perturbation to $G^*(x) = f^{*\prime}(x)$.

Let (f_n, G_n) be the nth image of (f_c, G) under the reduced renormalization transformation $\tilde{\mathcal{R}}$. In the case of $G(x) = \varepsilon\Phi(x)$, $G_n(x) \simeq \varepsilon\Phi_n(x)$, while $G_n(x) \simeq \varepsilon[f_n(x) + \Phi_n(x)]$ for the case of $G(x) = f_c(x) + \varepsilon\Phi(x)$. Here, $\Phi_n(x)$ is the nth image of Φ under the reduced linearized operator $\tilde{\mathcal{L}}$ of Eq. (25). For large n, it becomes

$$\Phi_n(x) \simeq \sum_{k=0}^{z-2} \alpha_k \nu_k^n \Phi_k^*(x) + \alpha_{z-1} p^n f^{*\prime}(x) \quad (31)$$

because the irrelevant part of Φ_n becomes negligibly small for large n.

The stability multipliers $\lambda_{1,n}$ and $\lambda_{2,n}$ of the p^n-periodic orbit are the same as those of the fixed point of the n times renormalized map $\mathcal{N}^n(T)$, which are given by

$$\lambda_{1,n} = f_n'(\hat{x}_n), \quad \lambda_{2,n} = f_n'(\hat{x}_n) - 2G_n(\hat{x}_n). \quad (32)$$

Here, $\hat{x}_n$ is just the fixed point of $f_n(x)$ [*i.e.*, $\hat{x}_n = f_n(\hat{x}_n)$] and converges to the fixed point $\hat{x}$ of the 1D fixed map $f^*(x)$ as $n \to \infty$. The first stability multiplier $\lambda_{1,n}$ converges to the 1D critical stability multiplier $\lambda^* = f^{*\prime}(\hat{x})$ as $n \to \infty$. For infinitesimally small ε, $\lambda_{2,n}$ has the form

$$\begin{aligned}\lambda_{2,n} &\simeq \pm\lambda_{1,n} - 2\varepsilon\Phi_n \\ &\simeq \pm\lambda^* + \varepsilon\left[\sum_{k=0}^{z-2} e_k \nu_k^n + e_{z-1} p^n\right] \text{ for large } n, \quad (33)\end{aligned}$$

where the plus and minus signs in front of $\lambda_{1,n}$ and λ^* correspond to the case of $G(x) = \varepsilon\Phi(x)$ and the case of $G(x) = f_c(x) + \varepsilon\Phi(x)$, respectively, and $e_k = -2\alpha_k \Phi_k^*(\hat{x})$ $(k = 0, \ldots, z-2)$ and $e_{z-1} = -2\alpha_{z-1} f^{*\prime}(\hat{x})$. Therefore, the slope S_n of $\lambda_{2,n}$ at the critical point ($\varepsilon = 0$) is

$$S_n \equiv \left.\frac{\partial\lambda_{2,n}}{\partial\varepsilon}\right|_{\varepsilon=0} \simeq \sum_{k=0}^{z-2} e_k \nu_k^n + e_{z-1} p^n \text{ for large } n. (34)$$

Here, the coefficients $\{e_k \; ; \; k = 0, \ldots, z-1\}$ depend on the initial reduced function $\Phi(x)$ because the α_k's are determined only by $\Phi(x)$. Note that the magnitude of the slope S_n increases with n unless all the e_k's $(k = 0, \ldots, z-1)$ are zero.

We choose monomials x^l $(l = 0, 1, 2, \ldots)$ as the initial reduced coupling perturbations $\Phi(x)$ because any smooth function $\Phi(x)$ can be represented as a linear combination of monomials by a Taylor series. Expressing $\Phi(x) = x^l$ as a linear combination of eigenfunctions of $\tilde{\mathcal{L}}$, we have

$$\begin{aligned}\Phi(x) = x^l &= \alpha_l \Phi_l^*(x) + \alpha_{z-1} f^{*\prime}(x) \\ &\quad + \sum_{n=1}^{\infty} \beta_n f^{*\prime}(x)[f^{*n}(x) - x^n] \quad (35)\end{aligned}$$

where α_l is nonzero for $l < z-1$ and zero for $l \geq z-1$, and all β_n's are irrelevant components. Note that two relevant components α_l and α_{z-1} exist for $l < z-1$, while only one relevant component α_{z-1} exists for $l \geq z-1$. The growth of the slope S_n for sufficiently large n is governed by the largest CE $\nu_{\max}$:

Table 1. The sequences $\{r_n\}$ for the one-term scaling law are shown when $\Phi(x) = 1$, x, x^2, and x^3.

n	$\Phi(x) = 1$	$\Phi(x) = x$	$\Phi(x) = x^2$	$\Phi(x) = x^3$
1	-30.099	9.986	-3.007	2.971 97
2	-31.559	10.09	-3.233	3.005 25
3	-31.278	9.955	-3.086	2.999 02
4	-31.328	9.947	-3.214	3.000 18
5	-31.319	9.938	-3.095	2.999 97
6	-31.320	9.937	-3.208	3.000 01

Table 2. Two sequences $\{r_{1,n}\}$ and $\{r_{2,n}\}$ for the two-term scaling law are shown when $\Phi(x) = x^2$.

n	$r_{1,n}$	$r_{2,n}$
1	-3.142 8	2.01
2	-3.153 9	2.75
3	-3.151 7	2.94
4	-3.152 2	2.99

$$S_n \sim \nu_{\max}^n. \tag{36}$$

For $l \geq z-1$, $\nu_{\max}$ is always p (*i.e.*, $\nu_{\max} = p$), while for $l < z-1$, $\nu_{\max}$ is the larger one between the two CE's ν_l and p (*i.e.*, if $|\nu_l| > p$ then $\nu_{\max} = \nu_l$; otherwise, $\nu_{\max} = p$).

Taking the quartic-maximum ($z = 4$) case as an example, we numerically study the growth of the slopes S_n's for the period-tripling ($p = 3$) case and confirm the one-term scaling law (36). We follow the periodic orbits of period 3^n up to level $n = 7$ and obtain the slopes S_n at the zero-coupling critical point $(A_\infty^{(3)}, 0)$ ($A_\infty^{(3)} = 1.909\,335\,470\,794\,655\ldots$) when the reduced coupling function $\Phi(x)$ is a monomial x^l ($l = 0, 1, 2, \ldots$). Since the magnitude of α ($\alpha = -3.152\ldots$) is larger than 3, $\nu_{\max} = \alpha^{(3-l)}$ for $l = 0$, 1, 2.

We define the growth rate of the slopes as follows:

$$r_n \equiv \frac{S_{n+1}}{S_n}. \tag{37}$$

Then, it will converge to a constant r ($= \nu_{\max}$) as $n \to \infty$. Four sequences of $\{r_n\}$ for $\Phi(x) = x^l$ ($l = 0, 1, 2, 3$) are shown in Table 1. It seems that they converge to their limit values, $r = \alpha^3$, α^2, α, and 3, respectively. However, the sequence for the case $\Phi(x) = x^2$ slowly converges to its limit value $r = \alpha$, as compared with the other three cases. This is because the value of the second relevant CE $\nu = 3$ for this case is close to that of $|\alpha|$. In order to see better convergence, the effect of the CE $\nu = 3$ must be taken into account. Then, the sequence obeys a two-term scaling law [17],

$$S_n = c_1 r_1^n + c_2 r_2^n, \quad \text{for large } n, \tag{38}$$

where c_1 and c_2 are some constants. Two sequences $\{r_{1,n}\}$ and $\{r_{2,n}\}$ are shown in Table 2. They seem to converge to their limit values $r_1 = \alpha$ and $r_2 = 3$. Note that the accuracy of r_1 ($= \alpha$) is better than that of r ($= \alpha$) obtained above by the one-term scaling analysis.

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