Routing Patterns for Periodic Orbits in the Coupled Standard Map

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Area-preserving maps have wide applications in physical problems and thus have been studied extensively.[1,2] But higher dimensional symplectic maps also occur in many physical problems.[1-7] Therefore, it is necessary to study their properties. To study orbits that have rotation vectors, first the periodic orbits must be located carefully. For example, by using the routing patterns for periodic orbits found by Shenker and Kadanoff,[8] not only the periodic orbits but also quasi-periodic orbits in the standard map can be located well. In this short note, we study periodic orbits in a coupled standard map that is a 4-dimensional symplectic map and report the routing patterns for them.

The map T that we study is a coupled standard map[3]:

$$T: \begin{cases} p_1' = p_1 - k_1/2 \cdot \sin(2\pi x_1) - c/2\pi \cdot \sin(2\pi(x_1+x_2)), \\ p_2' = p_2 - k_2/2 \cdot \sin(2\pi x_2) - c/2\pi \cdot \sin(2\pi(x_1+x_2)), \\ x_1' = x_1 + p_1', \\ x_2' = x_2 + p_2', \end{cases}$$

$$Z = (p_1, p_2, x_1, x_2) \in R^4.$$

Here, k_1 and k_2 are nonlinear parameters and c is a coupling parameter. Since the map T is periodic in p_1, p_2, x_1 and x_2 with a unit period, the phase space of this map can be reduced to the 4-torus(T^4). Since the set of orbit points lies on the 4-torus, escape cannot occur. This facilitates the numerical study.

First, we want to mention some properties of this map. The map T is symplectic since the linearized map M of T satisfies

$$M^t \cdot J \cdot M = J,\ J = \begin{bmatrix} 0 & -E \\ E & 0 \end{bmatrix},$$

where E is the 2×2 identity matrix and M^t the transpose of M.

Furthermore, the map has a reversible symmetry since it can be factorized into the product $(TS)\cdot S$ of two involutions:

$$S^2 = I = (TS)^2,\ I\text{: identity map},$$

$$S: \begin{cases} p_1' = p_1 - k_1/2\pi \cdot \sin(2\pi x_1) - c/2\pi \cdot \sin(2\pi(x_1+x_2)), \\ p_2' = p_2 - k_2/2\pi \cdot \sin(2\pi x_2) - c/2\pi \cdot \sin(2\pi(x_1+x_2)), \\ x_1' = -x_1 + n_1, \\ x_2' = -x_2 + n_2, \end{cases}$$

$$(n_1, n_2) \in Z^2.$$

The set of fixed points of the involutions form planes called symmetry planes. There are 8 symmetry planes as shown in Table 1.

Secondly, we want to mention the rotation vector of an orbit in this map. An orbit is said to have a rotation vector (ω_1, ω_2) if the average number of rotations per iteration of the map in both x_1- and x_2- directions exist:

$$\omega_1 = \lim_{n\to\infty} (x_1^{(n)} - x_1^{(0)})/n,$$

$$\omega_2 = \lim_{n\to\infty} (x_2^{(n)} - x_2^{(0)})/n,$$

$$Z^{(n)} = T^{(n)}(Z^{(0)}),\ Z = (p_1, p_2, x_1, x_2),$$

$T^{(n)}$: the nth power of T.

Orbits that have rotation vectors can be classified into 3 classes according to the rational dependence between ω_1 and ω_2. When there is no rational dependence, the orbit exhibits a quasi-periodic motion with 3 fundamental frequencies and lies on a 2-torus(T^2). On the other hand, the orbit exhibits a quasi-periodic motion with 2 fundamental frequencies or periodic motion according as there is a single or 2 independent rational relations. It is very difficult to locate a quasi-periodic orbit with 2- or 3-frequencies directly. But the quasi-periodic orbit can be approached indirectly by following the periodic orbits whose rotation vectors correspond to convergents to the rotation vector of the quasi-periodic orbit. Therefore, by locating periodic orbits, not only periodic orbits but also quasi-periodic orbits can be studied.

In general, to locate a periodic orbit, it is necessary to search the entire 4-dimensional space. But the coupled

Table 1. Eight symmetry planes in the coupled standard map.

symmetry plane	(x_1, x_2)
I	(0, 0)
II	(1/2, 1/2)
III	(0, 1/2)
IV	(1/2, 0)
V	$(p_1/2, p_2/2)$
VI	$((p_1+1)/2, (p_2+1)/2)$
VII	$(p_1/2, (p_2+1)/2)$
VIII	$((p_1+1)/2, p_2/2)$

standard map has a reversible symmetry which greatly simplifies this task. Reversibility implies that if $\{Z^{(n)}, n \in Z\}$ is an orbit of the map T, then $S \cdot \{Z^{(n)}, n \in Z\}$ is an orbit of its time reversal $T^{(-1)}$. A symmetric orbit is one that is invariant under S, i.e. an orbit that is its own time reversal. It is easy to show that a symmetric orbit must have a point on some symmetry plane and if the initial point of the orbit lies on some symmetry plane, then the half-way point around the orbit also lies on another symmetry plane. Therefore, to locate the symmetric periodic orbits, it is sufficient to search only a 2-dimensional symmetry plane, which reduces the original 4-dimensional search to a 2-dimensional search. Furthermore, a symmetric periodic point can be located by going only halfway around the orbit.

We found numerically that there are four kinds of symmetric periodic orbits with the same rotation vector $(p/r, q/r)$ that persist when the map parameters go to zero. Of course, a symmetric periodic orbit with the same rotation vector may be born by a tangent bifurcation. But, this orbit does not persist when the map parameters go to zero. Furthermore, we found a relation between these symmetric periodic orbits and the symmetry planes which is called the routing pattern for symmetric periodic orbits. There are 7 routing patterns which are determined by whether p,q and r are each even or odd. The result is summarized in Tables 2 and 3. The initial point $(Z^{(0)})$ of the orbit lies on the initial plane and the half-way point ($Z^{(r/2)}$ for even r, $Z^{((r+1)/2)}$ for odd r) on the final plane. Using these routing patterns, the four symmetric periodic orbits denoted by CL1, CL2, CL3 and CL4 in Tables 2 and 3 can be located easily, and thus not only the periodic orbits but also quasi-periodic orbits can be located easily.

It is remarkable that symmetric periodic orbits with

Table 2. Routing patterns for symmetric periodic orbits with even r. O denotes an odd number and E an even number.

(p,q,r)		initial plane	final plane
(O,O,E)	CL1:	I	II
	CL2:	III	IV
	CL3:	V	VI
	CL4:	VII	VIII
(E,O,E)	CL1:	I	III
	CL2:	II	IV
	CL3:	V	VII
	CL4:	VI	VIII
(O,E,E)	CL1:	I	IV
	CL2:	II	III
	CL3:	V	VIII
	CL4:	VI	VII

Table 3. Routing patterns for symmetric periodic orbits with odd r. O denotes an odd number and E an even number.

(p,q,r)		initial plane	final plane
(O,O,O)	CL1:	I	VI
	CL2:	II	V
	CL3:	III	VIII
	CL4:	IV	VII
(E,O,O)	CL1:	I	VII
	CL2:	II	VIII
	CL3:	III	V
	CL4:	IV	VI
(O,E,O)	CL1:	I	VIII
	CL2:	II	VII
	CL3:	III	VI
	CL4:	IV	V
(E,E,O)	CL1:	I	V
	CL2:	II	VI
	CL3:	III	VII
	CL4:	IV	VIII

positive residues have one point on the so-called dominant symmetry line in the 2-dimensional standard map.[8] A noble critical invariant circle exhibits a 'well-behaved' scaling behavior near the accumulation point on the dominant symmetry line.[8] But such a dominant symmetry plane is not observed in the coupled standard map. Therefore, it is not expected that there will be a 'well-behaved' scaling behavior for any quasi-periodic orbit.

To sum up, there are 7 routing patterns for symmetric periodic orbits and thus, using them, quasi-periodic orbits as well as periodic orbits can be located easily.

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결합된 표준 본뜨기에서 주기궤도들의 행로

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4차원 해밀토니안 본뜨기인 결합된 표준 본뜨기에서 주기궤도들을 공부했다. 주어진 회전벡터에 대해서 4개의 대칭적인 주기궤도들이 관측되었다. 이 대칭적인 주기궤도들과 8개의 대칭평면들 사이에는 7가지의 행로가 관측되었다. 일반적으로, 준주기궤도들은 주기궤도들로 점차적으로 접근해 갈 수 있다. 따라서, 이 7가지 행로에 의해서 주기궤도 뿐만 아니라 준주기궤도도 쉽게 찾아낼 수 있다.