Critical Phenomena for Period n-tuplings in 4-dimensional Volume-preserving Maps

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We study period-1 scaling behaviors for period n-tuplings in four-dimensional volume-preserving maps when $n = 2$. We find that there are four fundamental noncoordinate scaling factors, δ_1 and δ_2 (divergence rates from the period-1 map) and δ'_1 and δ'_2 (convergence rates to the period-1 map). Therefore, the parameter scaling factors, γ_1 and γ_2, for any bifurcation path are some combination of the fundamental noncoordinate scaling factors. We obtain the four fundamental noncoordinate scaling factors by a direct numerical study and find that there are three kinds of period-1 scaling behaviors.

I. INTRODUCTION

The discovery of universal scaling behaviors for period doublings in one-dimensional (1-D) maps by Feigenbaum[1] inspired several people[3-10] to study scaling behaviors for period doublings in area-preserving maps. Although the universality results for period doublings in 1-D maps extend to higher dimensional dissipative maps[2], the universal scaling behaviors in area-preserving maps are distinctly different from those in dissipative maps[3-10]. An interesting question is whether the self-similar period-doubling patterns of area-preserving maps carry over to higher dimensional conservative maps. Therefore, period doublings in four-dimensional (4-D) symplectic maps have recently been studied[10-12]. However, clear evidence for an infinite period-doubling sequence in a 4-D symplectic map has been reported by Mao, Satija and Hu[13]. The infinite period doubling sequence was determined by following a special bifurcation path. Many additional bifurcation paths and their scaling behaviors in a symmetric 4-D volume-preserving map have been found by a direct numerical method[14] and a renormalization method[15]. However, these authors[13-15] did not obtain the fundamental noncoordinate scaling factors.

By generalizing the bifurcation routes and the bifurcation paths introduced in Ref. 14, we find that there are infinite kinds of bifurcation routes. In this paper, we study among them only the 'period-1' scaling behaviors in the 'period-1' bifurcation routes. We find that there are three kinds of 'period-1' bifurcation routes. Each bifurcation route is characterized by its own four fundamental noncoordinate scaling factors, δ_1 and δ_2 (divergence rates from the period-1 map of the renormalization transformation) and δ'_1 and δ'_2 (convergence rates to the period-1 map). Therefore, the two parameter scaling factors γ_1 and γ_2 for any (regular or special) bifurcation path in a bifurcation route are some combination of the four fundamental noncoordinate scaling factors. We obtain these four fundamental noncoordinate scaling factors by a direct numerical method. Although the 'period-1' scaling behaviors were studied[13-15], the fundamental noncoordinate scaling factors were not found and only one special path for each of the bifurcation routes (L-, U- and E-route) was found, whereas we have found three kinds of special paths for our S- and A-routes and two kinds of special paths for our E-route.

This paper is organized as follows. We begin by collecting some useful properties of a symmetric 4-D volume-preserving map in Sec. II. We then generalize the bifurcation routes and the bifurcation paths defined in Ref. 14. In sec. III, the results of 'period-1' scaling behaviors are given and the final Section IV is a summary.

II. BIFURCATION ROUTES AND BIFURCATION PATHS

We present some useful properties of a symmetric 4-D volume-preserving map in subsection 1. We then generalize in subsections 2 and 3 the bifurcation routes and bifurcation paths defined in Ref. 14.

1. Symmetric 4-D Volume-Preserving Maps

We study the period-1 scaling behaviors in a symmetric 4-D volume-preserving map which was first studied in Ref. 14. The symmetric 4-D volume-preserving map T is of the following form:

$$T: \begin{cases} x' = -y - f(x, u), \\ y' = x, \\ u' = -v - g(x, u), \\ v' = u, \end{cases} \qquad (2.1)$$

where $f(x, u) = 2[(Cx + x^2) + E(u + Fu^2 - Gxu)]$ and $g(x, u) = f(u, x)$. The term $(Cx + x^2)$ in $f(x, u)$ is a quadratic function of x, and C a parameter. The term $(u - Fu^2 - Gxu)$ in $f(x, u)$ contains all the coupling terms up to quadratic terms. E is their common coefficient, called the coupling parameter. F and G are parameters, but we will fix their values to perform a two-parameter search[13]. Therefore, the map T is a symmetrically coupled quadratic Henon map.

There are two kinds of orbits in map (2.1)[14]. One is the in-phase orbit:

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, \ldots, N, \text{ where } N \text{ is the period.}$$

The other one is the opposite-phase orbit:

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} x_{i+N/2} \\ y_{i+N/2} \end{pmatrix}, \text{ where } N \text{ is the period.}$$

In this paper, we consider only the in-phase orbit.

For the in-phase orbit, the Jacobian matrix L of map (2.1) decomposes into two 2×2 matrices under a coordinate change[14]. By introducing new coordinates (X, Y, U, V) defined by

$$X = (x + u) \cdot R/2, \quad Y = (y + v) \cdot R/2,$$
$$U = (x - v) \cdot R/2, \quad V = (y - V) \cdot R/2 \qquad (2.2)$$

where $R = 1 + EF + EG$,

the old map T(2.1) becomes a new map T:

$$T: \begin{cases} X' = -Y - F(X, U), \\ Y' = X, \\ U' = -V - G(X, U), \\ V' = U, \end{cases} \qquad (2.3)$$

where

$F(X, U) = 2(PX - X^2 + F_1 U^2)$,

$G(X, U) = 2[(P - 2E)U + G_1 XU]$, and

$P = C - E$, $F_1 = (1 + EF - EG)/R$, $G_1 = 2(1 - EH)$,

$H = (2F + G)/R$.

Then the in-phase orbit of the old map (2.1) becomes the orbit of the new map with $U = V = 0$. Moreover, the values of two new coordinates (X, Y) can be determined by the 2-D Henon map,

$$X' = -Y + 2(PX + X^2), \quad Y' = X. \qquad (2.4)$$

The Jacobian matrix L of the new map at the in-phase orbit is decomposed:

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \qquad (2.5)$$

where 0 is the 2×2 null matrix,

and $$L_1 = \begin{pmatrix} 2P + 4X & -1 \\ 1 & 0 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 2(P - 2E) + 2G_1 X & -1 \\ 1 & 0 \end{pmatrix}.$$

Here, the matrix L_1 is just the Jacobian matrix of the 2-D Henon map (2.3).

Map T (2.1) is symplectic[14] only if

$$\frac{\partial f}{\partial u} = \frac{\partial g}{\partial x}. \qquad (2.6)$$

The stability of an orbit of period N in a 4-D symplectic map is determined by the Jacobian matrix M of T^N which is symplectic. As is well known[16], if λ is an eigenvalue of M, then so are λ^{-1} and λ^* (complex conjugate of λ). Therefore, the eigenvalues, λ's, come either in reciprocal pairs which are real, or in a complex quadruplet with $\lambda_1 = \lambda_2^{-1} = \lambda_3^* = \lambda_4^{*-1}$. These eigenvalues of M are called the multipliers of the orbit[10]. Following Broucke[17], Howard and Mackay[18] associate with each eigenvalue λ a stability index,

$$\rho = \lambda + \lambda^{-1}. \qquad (2.7)$$

Then, the reduced characteristic polynomial of a 4-D symplectic matrix is quadratic[18]:

$$\rho^2 - T_1 \cdot \rho + T_2 - 2 = 0. \qquad (2.8)$$

where

$$T_1 = \sum_{i=1}^{4} \lambda_i = TrM = \rho_1 + \rho_2,$$

$$T_2 = \sum_{\substack{i,j \\ i<j}} \lambda_i \cdot \lambda_j = [(TrM)^2 - Tr(M^2)]/2 = \rho_1 \cdot \rho_2 + 2.$$

Therefore, the two independent quantities (T_1, T_2) or (ρ_1, ρ_2) determine the stability of the periodic orbit.[17,18] A periodic orbit is spectrally stable if all stability indices are real with $|\rho| \le 2$ and a period-doubling bifurcation occurs when two eigenvalues coalease at $\lambda = -1$ and split along the negative real axis (a stability index decreases through -2)[18].

The map T (2.1) is a volume-preserving map, since Det(L)= 1 and it is symplectic only if $G = 2F$. However, for the in-phase orbit, Eq. (2.6) is always satisfied because of the symmetry of the map $(g(x, u) = f(u, x))$. Therefore, the stability diagram in the T_1- T_2 plane for the orbits in a 4-D symplectic map is the same as that for the in-phase orbit in a symmetric 4-D volume preserving map[14]. Since the Jacobian matrix of the new map (2.3) at the in-phase orbit is decomposed as shown in Eq. (2.5), the stability index ρ_1 of a periodic orbit is a function of only one parameter P and ρ_2 a function of two parameters P and E, since we fix the values of F and G to perform a two-parameter search[13]:

$$\rho_1 = \rho_1(P) \text{ and } \rho_2 = \rho_2(P, E). \qquad (2.9)$$

2. Bifurcation Routes

A remarkable observation of Mao and Helleman[14] is that a mother stability region bifurcates into two daughter stability regions in the parameter plane as shown in Fig. 1. Therefore, the stability diagram in the parameter plane can be regarded as a kind of binary tree. We denote the upper branch of the two daughter stability regions by the letter 'U' and the lower branch by the letter 'L'. Then, a bifurcation route is uniquely determined by its address which is an infinite sequence of the two letters 'U' and 'L' as shown in Fig. 1. Therefore, there are infinite kinds of bifurcation routes. In this paper, we

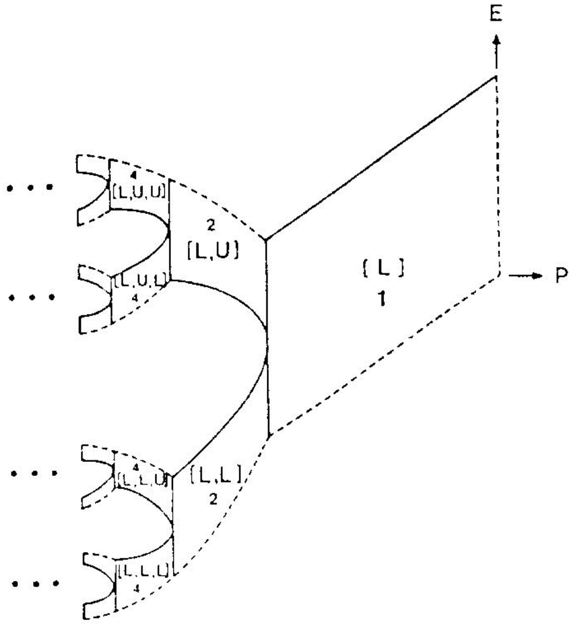

Fig. 1. A skematic stability diagram in the PE-parameter plane for period-1,-2,-4, etc., orbits of map (2.3). The period-doubling bifurcation line is denoted by the solid line, and the tangent bifurcation line is denoted by the dashed line.

consider only the 'period-1' bifurcation routes among them.

We find that there are three kinds of 'period-1' bifurcation routes. The three kinds of 'period-1' bifurcation routes are as follows. The first one is the 'S-route' whose address is $[a, (U,)^\infty]$ or $[b, (L,)^\infty]$, where a and b are arbitrary finite sequences. That is, an 'S-route' is formed if one follows asymptotically only the upper branches or only the lower branches. Since one goes asymptotically in the same direction ('U'- or 'L'-direction) in the 'S-route', we call it the 'S-route'. The second one is the 'A'-route whose address is $[c, (L,U,)^\infty]$, where c is an arbitrary finite nonempty sequence. Since the address of an 'A-route' is $[c, (L,U,)^\infty]$, the direction of the route asymptotically alternates between the 'L'-direction and 'U'-direction. Therefore, we call it the 'A-route'. The third one is the 'E-route' whose address is $[(L,U,)^\infty]$. Since the address is unique, there is only one 'E-route'. The difference between an 'A-route' and the 'E-route' is as follows. The value of the coupling parameter E^* at the accumulation point (P^*, E^*) for period doublings in the 'E-route' is zero, whereas the value of E^* in any 'A-route' is nonzero. In these three kinds of 'period-1' bifurcation routes, the period-doubling patterns exhibit their respective 'period-1' scaling behaviors (see Sec. III).

3. Bifurcation Paths

In this subsection, we define the 'period-1' bifurcation paths and compare them with those previously defined in Ref. 14.

Before defining the bifurcation paths, we explain some terms and notations which will be used later. We call an orbit born by the nth period-doubling bifurcation in map (2.3) an orbit of level n. Then, the period N of an orbit of level n is 2^n and there are 2^n orbits of level n. As explained in subsection A, the stability of an orbit of level n is determined by its multipliers $(\lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}, \lambda_{4,n})$ or its stability indices $(\rho_{1,n}, \rho_{2,n})$ or two independent quantities $(T_{1,n}, T_{2,n})$ defined in Eq. (2.8). The values of the two parameters P and E in map (2.3) for which an orbit of level n has some given multipliers or equivalently some given stability indices will be denoted by P_n and E_n.

Let us choose a 'period-1' bifurcation route. Then, a 'period-1' bifurcation path which belongs to the chosen bifurcation route is formed by following in the chosen bifurcation route using P_n and E_n for which the orbit of level n has some given multipliers (λ_1, λ_1^{-1}, $e^{i\theta}$, $e^{-i\theta}$) or equivalently some given stability indices (ρ_1, ρ_2) where

$$\lambda_1 \text{ is any real number } (\lambda_1 \in R),\ 0 \leqq \theta \leqq \pi \quad (2.10)$$
$$\rho_1 \in R \text{ and } |\rho_2 = 2\cdot\cos\theta| \leqq 2.$$

The 'period-1' scaling behaviors have been studied previously[14] and three kinds of bifurcation routes were found (L-route, U-route, and E-route). We compare our 'period-1' bifurcation routes with those found by Mao and Helleman. They first defined bifurcation paths. Their bifurcation paths are formed by following asymptotically P_n and E_n for which the orbit of level n has some given multipliers (-1, -1, $e^{i\theta}$, $e^{-i\theta}$) which correspond to a bifurcation point lying on the period-doubling bifurcation line in the T_1-T_2 plane. A bifurcation path with $0 \leq \theta \leq \pi/2$ is called an L_θ path (or L path if θ is not specified)[14]. Then, an L-route is formed by a paticular L path and all L_θ paths in its neighborhood and its address is $[a,(U,)^\infty]$ or $[b,(L,)^\infty]$[14] which is the same as that of our 'S-route'. However, they considered only the case for $\lambda_1 = -1$ and $0 \leq \theta \leq \pi/2$, whereas we consider the case for $\lambda_1 \in R$ and $0 \leq \theta \leq \pi$ for the 'S-route' (see Eq. (2.10)). In such a sense, their L-route is a proper subset of our 'S-route', Similarly, it is easy to show that their U-route and E-route are proper subsets of our 'A-route' and 'E-route', respectively. By generalizing the 'period-1' bifurcation routes and 'period-1' bifurcation paths as mentioned above, we find in our 'period-1' bifurcation routes that there are more 'special' bifurcation paths than those found in Ref. 14 (see Sec. III).

III. PERIOD-1 SCALING BEHAVIORS

In this section, the effects of 'period-1' scaling behaviors in the three kinds of 'period-1' bifurcation routes (S-, A-, and E-route) are given. In subsection A, we obtain the two parameter scaling factors, γ_1 and γ_2, by the scaling matrix method[19]. The values of γ_1 and γ_2 depend on the 'period-1' bifurcation paths. We find that there are more special bifurcation paths than those found in Ref. 14. Furthermore, we find in subsection B the four fundamental noncoordinate scaling factors, δ_1 and δ_2 (divergence rates from the fixed map of the renormalization transformation) and δ_1' and δ_2' (convergence rates to the fixed map). Therefore, the parameter scaling factors γ_1 and γ_2 for any (regular or special) bifurcation paths in a 'period-1' bifurcation route are some combination of the four fundamental noncoordinate scaling factors of the bifurcation route. In the final subsection 3, we review the orbital scaling behaviors which have been studied in Ref. 14.

1. Parameter Scaling Factors

To perform a two-parameter search[13], we consider the case where the values of (F,G) are (1,2), (2,4), (1,3) and (2,3) and follow with quadruple precision the orbit with n level up to 17. The parameter scaling factors are independent of the values of F and G within numerical accuracy. This is expected as we are considering a codimension-two problem.

We first define regular paths and special paths as follows. Choose a 'period-1' bifurcation route. Then, for any 'period-1' bifurcation path which belongs to the chosen bifurcation route, (P_n, E_n) converges to the same accumulation point (P^*, E^*) in the chosen bifurcation route:

Table 1. The critical stability indices ρ_1 and ρ_2 in the 'period-1' bifurcation routes.

Route	ρ_1^*	ρ_2^*
S-route	-2.54351020	2.00000000
A-route	-2.54351020	-1.00000000
E-route	-2.54351020	-2.54351020

$$\lim_{n\to\infty}(P_n, E_n) = (P^*, E^*) \text{ for all bifurcation paths.} \tag{3.1}$$

Furthermore, at the accumulation point (P^*, E^*), the stability indices $\rho_{1,n}$ and $\rho_{2,n}$ converge geometrically to the critical stability indices ρ_1^* and ρ_2^*, respectively:

$$\lim_{n\to\infty}\rho_{1,n}(P^*)=\rho_1^* \text{ and } \lim_{n\to\infty}\rho_{2,n}(P^*, E^*)=\rho_2^*. \tag{3.2}$$

Note that $\rho_{1,n}$ is a function of only one parameter P(see Eq. (2.9)). The critical stability indicies are shown in Table 1. If the given values of the stability indices ρ_1 and ρ_2 in Eq. (2.10) are not the critical values ($\rho_1 \neq \rho_1^*$ and $\rho_2 \neq \rho_2^*$), then we call it a regular path, otherwise we call it a special path.

The scaling behavior of the period-doubling sequence $[(P_n, E_n), n = 0,1,2,...]$ can be determined by the scaling matrix method developed by Gukenheimer, Hu and Rudnick[19] (refer to Ref. 13 for details). The 2×2 scaling matrix of level n, Γ_n, is defined as follows:

$$\begin{pmatrix} P_n - P_{n-1} \\ E_n - E_{n-1} \end{pmatrix} = \Gamma_n \cdot \begin{pmatrix} P_{n+1} - P_n \\ E_{n+1} - E_n \end{pmatrix}. \tag{3.3}$$

Then, Γ_n approaches a constant matrix Γ as $n\to\infty$:

$$\lim_{n\to\infty}\Gamma_n = \Gamma. \tag{3.4}$$

The eigenvalues of Γ, γ_1 and γ_2, are the parameter scaling factors. The parameter scaling factors in the three kinds of 'period-1' bifurcation routes are shown in Table 2. The values of γ_1 and γ_2 in each bifurcation route depend on the bifurcation paths. Regular paths have the same values of γ_1 and γ_2, whereas each kind of special path has different values of γ_1 and γ_2 from those of the regular paths. In the S- and A-routes, there are three kinds of special paths. On the other hand, in the E-route, there are two kinds of special paths, since ρ_2 for any bifurcation path (see the range of ρ_2 in Eq. (2.10)) can not be ρ_2^* (see Table 1). No γ_2 exists for the 1st type of special path in the E-route, since E_n is zero for all n. In

Table 2. The parameter scaling factors, γ_1 and γ_2, for the 'period-1' bifurcation paths in the 'period-1' bifurcation routes. In the 2nd column, we denote regular paths by '*R*' and special paths by '*S*'. The ranges of ρ_1 and ρ_2 are given in Eq. (2.10).

Route	Path	γ_1	γ_2
S-route	$\rho_1 \neq \rho_1^*,\ \rho_2 \neq \rho_2^*$ (R)	8.721	4.000
	$\rho_1 \neq \rho_1^*,\ \rho_2 = \rho_2^*$ (S)	8.721	-15.08
	$\rho_1 = \rho_1^*,\ \rho_2 \neq \rho_2^*$ (S)	-74.78	4.000
	$\rho_1 = \rho_1^*,\ \rho_2 = \rho_2^*$ (S)	-74.78	-15.08
A-route	$\rho_1 \neq \rho_1^*,\ \rho_2 \neq \rho_2^*$ (R)	8.721	-2.000
	$\rho_1 \neq \rho_1^*,\ \rho_2 = \rho_2^*$ (S)	8.721	-15.08
	$\rho_1 = \rho_1^*,\ \rho_2 \neq \rho_2^*$ (S)	-74.78	-2.000
	$\rho_1 = \rho_1^*,\ \rho_2 = \rho_2^*$ (S)	-74.78	-15.08
E-route	$\rho_1 \neq \rho_1^*$ (R)	8.721	-4.404
	$\rho_1 = \rho_2 \in [-2, 2]$ (S)	8.721	non- existent
	$\rho_1 = \rho_1^*$ (S)	-74.78	-4.404

Ref. 14, they found only one special path in each bifurcation route (L-, U- and E-route) which belongs to the 1st type of special path in each bifurcation route (S-, A- and E-route). Therefore, by generalizing the bifurcation routes and the bifurcation paths as explained in Sec. II, we find that there are more special bifurcation paths than those found in Ref. 14.

2. Fundamental Noncoordinate Scaling Factors

In this subsection, we find that there are four fundamental noncoordinate scaling factors, δ_1 and δ_2 (divergence rates from the fixed map of the renormalization transformation) and δ_1' and δ_2' (convergence rates to the fixed map).

At the accumulation point (P^*, E^*) in a bifurcation route, the stability indices, $\rho_{1,n}(P^*)$ and $\rho_{2,n}(P^*, E^*)$ converge to the critical stability indices ρ_1^* and ρ_2^*, respectively. The convergence is asymptotically geometric at rates δ_1' and δ_2', respectively:

$$\rho_{1,n}(P^*) - \rho_1^* \sim \delta_1'^{\,n} \text{ and } \rho_{2,n}(P^*, E^*) - \rho_2^* \sim \delta_2'^{\,n}. \tag{3.5}$$

The values of δ_1' and δ_2' are shown in Table 3. Since $|\delta_2'| \geq |\delta_1'|$ (equality holds only for the E-route), δ_2' is the essential convergence rate as it is the largest noncoordinate eigenvalue inside the unit circle[10]. That is, a critical

Table 3. The four fundamental noncoordinate scaling factors, δ_1, δ_2, δ_1' and δ_2' in the 'period-1' bifurcation routes.

Route	δ_1	δ_2	δ_1'	δ_2'
S-route	8.721	4.000	-0.1166	-0.2653
A-route	8.721	-2.000	-0.1166	0.1326
E-route	8.721	-4.404	-0.1166	-0.1166

map on the critical map surface converges to the fixed map with rate δ_2' in the scaling coordinate[10].

First, we obtain the analytic formulae for δ_1 and δ_2 (divergence rates from the fixed map) by using the eigenvalue-matching renormalization method[20]. The basic idea of Derrida et al.[20] is to associate for each (P, E) a value (P', E') such that $T^{(n+1)}_{(P',E')}$ locally resembles $T^{(n)}_{(P,E)}$; $T^{(n)}$ is the 2^nth iterated map of T (i.e., $T^{(n)} = T^{2^n}$). An approximate way to do this is to equate the stability indices of level n $\rho_{1,n}(P,E)$ and $\rho_{2,n}(P, E)$ to those of level $(n+1)$, $\rho_{1,n+1}(P', E')$ and $\rho_{2,n+1}(P',E')$:

$$\rho_{1,n}(P, E) = \rho_{1,n+1}(P', E') \text{ and } \rho_{2,n}(P, E) = \rho_{2,n+1}(P', E'). \quad (3.6)$$

The accumulation point (P^*,E^*) is a fixed point of the recurrence relation (3.6):

$$\rho_{1,n}(P^*, E^*) = \rho_{1,n+1}(P^*, E^*),$$
$$\rho_{2,n}(P^*, E^*) = \rho_{2,n+1}(P^*, E^*). \quad (3.7)$$

By linearizing Eq. (3.7) about the accumulation point (P^*,E^*), we obtain

$$\begin{pmatrix} \Delta P \\ \Delta E \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial P'}|_* & \frac{\partial P}{\partial E'}|_* \\ \frac{\partial E}{\partial P'}|_* & \frac{\partial E}{\partial E'}|_* \end{pmatrix} \begin{pmatrix} \Delta P' \\ \Delta E' \end{pmatrix} \quad (3.8)$$
$$= \Delta_n \cdot \begin{pmatrix} \Delta P' \\ \Delta E' \end{pmatrix},$$

where

$$\Delta P = P - P^*,\ \Delta E = E - E^*,\ \Delta P' = P' - P^*,$$
$$\Delta E' = E' - E^*, \text{ and } \Delta_n = A_n \cdot B_n,$$

$$A_n = \begin{pmatrix} \frac{\partial \rho_{1,n}}{\partial P}|_* & \frac{\partial \rho_{1,n}}{\partial E}|_* \\ \frac{\partial \rho_{2,n}}{\partial P}|_* & \frac{\partial \rho_{2,n}}{\partial E}|_* \end{pmatrix},$$

$$B_n = \begin{pmatrix} \frac{\partial \rho_{1,n+1}}{\partial P'}|_* & \frac{\partial \rho_{1,n+1}}{\partial E'}|_* \\ \frac{\partial \rho_{2,n+1}}{\partial P'}|_* & \frac{\partial \rho_{2,n+1}}{\partial E'}|_* \end{pmatrix} \text{ and}$$

$*$ denotes the accumulation point (P^*,E^*).

Then, the eigenvalues $\delta_1^{(n)}$ and $\delta_2^{(n)}$ of the matrix Δ_n are

$$\delta^{(n)} = \frac{Tr\Delta_n \pm \sqrt{(Tr\Delta_n)^2 - 4 \cdot Det(\Delta_n)}}{2}. \quad (3.9)$$

Note that $\rho_1 = \rho_1(P)$ and $\rho_2 = \rho_2(P,E)$ in map (2.3) (see Eq. (2.9)). Therefore, after some algebra, we obtain the analytic formulae for $\delta_1^{(n)}$ and $\delta_2^{(n)}$:

$$\delta_1^{(n)} = \frac{d\rho_{1,n+1}/dP'|_*}{d\rho_{1,n}/dP|_*}, \quad (3.10)$$

$$\delta_2^{(n)} = \frac{\partial\rho_{1,n+1}/\partial E'|_*}{\partial\rho_{1,n}/\partial E|_*}.$$

As $n \to \infty$, $\delta_1^{(n)}$ and $\delta_2^{(n)}$ approach δ_1 and δ_2 which are the divergence rates from the fixed map:

$$\delta_i = \lim_{n\to\infty} \delta_i^{(n)}, \quad i = 1, 2. \quad (3.11)$$

Secondly, we obtain the analytic formulae for γ_1 and γ_2 (parameter scaling factors) by using the scaling matrix method[19]. Note that a bifurcaton path is formed by following in the chosen bifurcation route (P_n,E_n) at which the orbit of level n has some given stability indices ρ_1 and ρ_2 (see Eq. (2.10)). Let us denote the given stability indices ρ_1 and ρ_2 by $G\rho_1$ and $G\rho_2$ and write them in the following form:

$$G\rho_1 = \rho_1^* + \Delta\rho_1 \text{ and } G\rho_2 = \rho_2^* + \Delta\rho_2. \quad (3.12)$$

The, by the definition of a bifurcation path, we obtain

$$G\rho_1 = \rho_1^* + \Delta\rho_1 = \rho_{1,n}(P_n, E_n),$$
$$G\rho_1 = \rho_1^* - \Delta\rho_1 = \rho_{1,n+1}(P_{n+1}, E_{n+1}),$$
$$G\rho_2 = \rho_2^* + \Delta\rho_2 = \rho_{2,n}(P_n, E_n),$$
$$G\rho_2 = \rho_2^* - \Delta\rho_2 = \rho_{2,n+1}(P_{n+1}, E_{n+1}). \quad (3.13)$$

By linearizing Eq. (3.13) about the accumulation point (P^*,E^*) and using Eq. (3.5), after some algebra we obtain

$$\begin{pmatrix} \Delta P_n \\ \Delta E_n \end{pmatrix} = \Gamma_n \cdot \begin{pmatrix} \Delta P_{n+1} \\ \Delta E_{n+1} \end{pmatrix}, \quad (3.14)$$

where $\Delta P_n = P_n - P^*$, $\Delta E_n = E_n - E^*$, and $\Gamma_n = A_n^{-1} \cdot C_n$, where A_n is defined in Eq. (3.8), and C_n

depends on the values of $G\rho_1$ and $G\rho_2$ as follows:

1. $G\rho_1 \neq \rho_1^*$ and $G\rho_2 \neq \rho_2^*$ ($\Delta\rho_1 \neq 0$ and $\Delta\rho_2 \neq 0$), $C_n = B_n$ (B_n is defined in Eq. (3.8)).
2. $G\rho_1 \neq \rho_1^*$ and $G\rho_2 = \rho_2^*$ ($\Delta\rho_1 \neq 0$ and $\Delta\rho_2 = 0$),

$$C_n = \begin{pmatrix} \frac{\partial\rho_{1,n+1}}{\partial P'}\Big|_* & \frac{\partial\rho_{1,n+1}}{\partial E'}\Big|_* \\ \delta_2'^{-1}\cdot\frac{\partial\rho_{2,n+1}}{\partial P'}\Big|_* & \delta_2'^{-1}\cdot\frac{\partial\rho_{2,n+1}}{\partial E'}\Big|_* \end{pmatrix},$$

3. $G\rho_1 = \rho_1^*$ and $G\rho_2 \neq \rho_2^*$ ($\Delta\rho_1 = 0$ and $\Delta\rho_2 \neq 0$),

$$C_n = \begin{pmatrix} \delta_1'^{-1}\cdot\frac{\partial\rho_{1,n+1}}{\partial P'}\Big|_* & \delta_1'^{-1}\cdot\frac{\partial\rho_{1,n+1}}{\partial E'}\Big|_* \\ \frac{\partial\rho_{1,n-1}}{\partial P'}\Big|_* & \frac{\partial\rho_{1,n+1}}{\partial E'}\Big|_* \end{pmatrix},$$

4. $G\rho_1 = \rho_1^*$ and $G\rho_2 = \rho_2^*$ ($\Delta\rho_1 = 0$ and $\Delta\rho_2 = 0$),

$$C_n = \begin{pmatrix} \delta_1'^{-1}\cdot\frac{\partial\rho_{1,n+1}}{\partial P'}\Big|_* & \delta_1'^{-1}\cdot\frac{\partial\rho_{1,n+1}}{\partial E'}\Big|_* \\ \delta_2'^{-1}\cdot\frac{\partial\rho_{2,n-1}}{\partial P'}\Big|_* & \delta_2'^{-1}\cdot\frac{\partial\rho_{2,n+1}}{\partial E'}\Big|_* \end{pmatrix}.$$

Then, the eigenvalues $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ of the scaling matrix Γ_n are:

$$\gamma^{(n)} = \frac{Tr\Gamma_n \pm\sqrt{(Tr\Gamma_n)^2 - 4\cdot Det(\Gamma_n)}}{2}. \quad (3.15)$$

Note also that $\rho_1 = \rho_1(P)$ and $\rho_2(P,E)$ in map (2.3) (see Eq. (2.9)). Therefore, after some algebra, we obtain the analytic formulae for $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ which depend on the values of $G\rho_1$ and $G\rho_2$ as follows:

1. $G\rho_1 \neq \rho_1^*$ and $G\rho_2 \neq \rho_2^*$,

$$\gamma_1^{(n)} = \frac{d\rho_{1,n+1}/dP'|_*}{d\rho_{1,n}/dP|_*}, \quad \gamma_2^{(n)} = \frac{\partial\rho_{2,n+1}/\partial E'|_*}{\partial\rho_{2,n}/\partial E|_*}, \quad (3.16a)$$

2. $G\rho_1 \neq \rho_1^*$ and $G\rho_2 = \rho_2^*$,

$$\gamma_1^{(n)} = \frac{d\rho_{1,n+1}/dP'|_*}{d\rho_{1,n}/dP|_*}, \quad \gamma_2^{(n)} = \frac{\partial\rho_{2,n+1}/\partial E'|_*}{\partial\rho_{2,n}/\partial E|_*}\cdot\delta_2'^{-1}; \quad (3.16b)$$

3. $G\rho_1 = \rho_1^*$ and $G\rho_2 \neq \rho_2^*$,

$$\gamma_1^{(n)} = \frac{d\rho_{1,n+1}/dP'|_*}{d\rho_{1,n}/dP|_*}\cdot\delta_1'^{-1}, \quad \gamma_2^{(n)} = \frac{\partial\rho_{2,n+1}/\partial E'|_*}{\partial\rho_{2,n}/\partial E|_*}, \quad (3.16c)$$

4. $G\rho_1 = \rho_1^*$ and $G\rho_2 = \rho_2^*$,

$$\gamma_1^{(n)} = \frac{d\rho_{1,n+1}/dP'|_*}{d\rho_{1,n}/dP|_*}\cdot\delta_1'^{-1}, \quad \gamma_2^{(n)} = \frac{\partial\rho_{2,n+1}/\partial E'|_*}{\partial\rho_{2,n}/\partial E|_*}\cdot\delta_2'^{-1}. \quad (3.16d)$$

As $n\to\infty$, $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ approach γ_1 and γ_2 which are the parameter scaling factors:

$$\lim_{n\to\infty}\gamma_i^{(n)} = \gamma_i, \quad i=1,2 \quad (3.17)$$

By comparing the analytic formulae for $\delta_1^{(n)}$ and $\delta_2^{(n)}$ in Eq. (3.10) with the analytic formulae for $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ in Eq. (3.16), one can express the scaling factors γ_1 and γ_2 in terms of $\delta_1, \delta_2, \delta_1'$ and δ_2' as follows:

1. $G\rho_1 \neq \rho_1^*$ and $G\rho_2 \neq \rho_2^*$ (regular paths),
   $\gamma_1 = \delta_1$ and $\gamma_2 = \delta_2$, (3.18a)
2. $G\rho_1 \neq \rho_1^*$ and $G\rho_2 = \rho_2^*$ (the 1st type of special paths),
   $\gamma_1 = \delta_1$ and $\gamma_2 = \delta_2/\delta_2'$ (3.18b)
3. $G\rho_1 = \rho_1^*$ and $G\rho_2 \neq \rho_2^*$ (the 2nd type of special paths),
   $\gamma_1 = \delta_1/\delta_1'$ and $\gamma_2 = \delta_2$, (3.18c)
4. $G\rho_1 = \rho_1^*$ and $G\rho_2 = \rho_2^*$ (the 3rd type of special paths),
   $\gamma_1 = \delta_1/\delta_1'$ and $\gamma_2 = \delta_2/\delta_2'$. (3.18d)

Therefore, the fundamental noncoordinate scaling factors are δ_1 and δ_2 (divergence rates from the fixed map) and δ_1' and δ_2' (convergence rates to the fixed map) which are shown in Table 3. That is, the parameter scaling factors γ_1 and γ_2 for any (regular or special) bifurcation path are some combination of the four fundamental noncoordinate scaling factors. As shown in Table 3, the values of δ_1 and δ_1' are the same for all bifurcation routes and moreover, these values are the same as the values of δ and δ' for period-doubling in area-preserving maps (the value of δ' was found in Refs. 9 and 10). However, the values of δ_2 and δ_2' are different for the there kinds of bifurcation routes. Furthermore, since $|\delta_2'| \geq |\delta_1'|$ (equality holds only for the E-route), the essential convergence rate[10] is δ_2'. That is, δ_2' is the limiting factor in the convergence of a critical map on the critical map surface in the scaling coordinate[10].

3. Orbital Scaling Factors

In this subsection, we review the orbital scaling behaviors of the in-phase orbits found in Raf. 14 to contain in this paper all the scaling behaviors (parameter scaling behaviors and orbital scaling behaviors).

After making the linear transformation in Eq. (2.2), the old map (2.1) becomes the map (2.3). Then, the first two coordinates X and Y in the new map (2.3) of the in-phase orbit with $U = V = 0$ are determined by the 2-D Henon map (2.4). Therefore, X and Y scale with the 2-D orbital scaling factors[3-10] $\alpha = -4.018\cdots$, and $\beta = 16.36\cdots$. Furthermore, according to the definition of the linear transformation (2.2), the coordinates x and y (or u and v) of the in-phase orbit also scale with the same 2-D orbital scaling factors α and β.

IV. SUMMARY

By generalizing the bifurcation routes and the bifurcation paths defined in Ref. 14, we find that there are infinite kinds of bifurcation routes, whereas only three bifurcation routes (L-,U- and E-route) were found in Ref. 14. In this paper, we study among them only 'period-1' scaling behaviors in the 'period-1' bifurcation routes. It is shown that their L-,U- and E-routes are proper subsets of our 'period-1' bifurcation routes, S-,A- and E-route, respectively. Furthermore, we find that there are three kinds of special paths for our S- and A- routes and two kinds of special paths for our E-route, whereas only one special path each for the L-,U- and E- bifurcation routes was found.

The parameter scaling factors γ_1 and γ_2 depend on the bifurcation paths as shown in the Table 2. However, they are some combination of the four quantities δ_1, δ_2, δ'_1 and δ'_2 as shown in Eq. (3.18). Therefore, δ_1, δ_2, δ'_1 and δ'_2 are fundamental noncoordinate scaling factors. These results, Eq. (3.5) and Eq. (3.10) suggest that there exists a fixed map T^* of the renormalization transformation N for period-doubling and the linearized transformation DN of N at T^* has two unstable noncoordinate eigenvalues δ_1 and δ_2 (divergence rates from the fixed map) and two stable noncoordinate eigenvalues δ'_1 and δ'_2 (convergence rates to the fixed map). Finally, note that there are four fundamental noncoordinate scaling factors in 4-D volume-preserving maps, whereas there are only two fundamental noncoordinate scaling factors in area-preserving maps. This is expected as we are considering a codimension-two problem.

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4차원 보존 본뜨기에서 n 갈림의 임계현상

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4차원 보존 본뜨기에서 n 이 2일 때 n 갈림의 주기-1 축척거동을 공부했다. 4개의 기본적인 비좌표 축척인수들이 존재함을 알았다. 따라서, 어떤 두 갈림길의 매개변수 축척인수들도 이 4개의 기본적인 비좌표 축척인수들로 나타낼 수 있다. 이 기본적인 비좌표 축척인수들의 값을 수치적으로 구해냈으며, 3가지의 주기-1 축척거동이 있음을 찾아냈다.