Renormalization Analysis of m/n-Bifurcations and Invariant Curves in Area-Preserving Maps

Sang-Yoon Kim, Koo-Chul Lee

Department of Physics,
Seoul National University, Seoul 151

Duk-In Choi

Department of Physics, Korea Advanced Institute of Science and Technology,
P.O. Box 150, Chongyangri, Seoul

(Received 31 July 1986)

Renormalization analysis of m/n-bifurcation sequences is reported for 2-dimensional reversible area-preserving maps, generalizing the method of quadratic approximants. Quadratic approximants are formed by retaining upto quadratic terms in the Taylor expansion of the p-iterate of map T about the p-periodic point, T^p. Recurrence relations among coefficients of the expansion of two successive orders of the sequence are elements of the renormalization group. The bifurcation ratios and scaling factors calculated by this method agree well with the values estimated directly by following the sequences. This method can also be applied to study other infinitely nested self-similar structures. To show the point we apply this technique to the critical behavior of a noble invariant curve and obtain universal scaling constants.

I. INTRODUCTION

The motions of nonintegrable systems moderately perturbed from integrable ones can be divided generally into two types, regular and chaotic orbits. Regular components consists of main fixed points, vibrational invariant curves around the main fixed points and daughter island chains born out of resonance-bifurcation of the main fixed points. Chaotic orbits are interwoven with the island chains since hyperbolic points are also born out of resonance bifurcation and separatrices split generically. The daughter island chains have their own vibrational invariant curves and secondary island chains around them, and so on. In this way the island chains have infinitely nested structures[1].

In our previous work we have shown that at a certain parameter value island chains of every generation of a particular resonance do exist and they have self similar structure nested asymptotically. We found that their limiting behaviours are self-similar and calculated scaling factors for the 1/n-resonance sequences with a varying n from 3 to 6[2]. We have also observed that the pattern of periodic orbits repeat itself asymptotically from one bifurcation to the next for even n and to every other for odd n.

Recently it has been recognized that these self-similar scaling behaviors have paramount importance in constructing the transport equation in the divided phase space where regular and chaotic orbits coexist[3]. Stochastic orbits have long time correlations near islands and their scaling behaviors are directly related to the transition probabilities of a Markov tree model which describes the diffusion of particles.

In this paper we study these asymptotically self-similar island structures by a simple approximate renormalization method. The renormalization method was introduced into the dynamical system first by Feigenbaum[4] to study the Feigenbaum sequence of the 1-dim. dissipative map. The method was soon extended to 2-dim. area-preserving maps for the study of period doubling sequences and critical invariant curves. Collet et al.[5] and Widom and Kadanoff[6] solved directly the fixed point equation for the renormalization of 1/2-bifurcation in map- and action-space, respectively, and obtained an approximate fixed point and scaling factors. By linearizing the renormalization transformation about the fixed point, they obtained eigenvalues. Using MACSYMA, Greene et al.[7] also obtained an approximate universal map, but they used the accumulation parameter value and scaling factors obtained by directly following the 1/2-bifurcation sequence. The direct methods for solving the fixed point equation for the renormalization of 1/n-bifurcation becomes rapidly intractable as n increases. Therefore, for higher n-tupling bifurcation, it is desirable to use an approximate renormalization method in which the difficulty of calculations does not increase significantly with n. Recently Lichtenberg[8] obtained accumulation points for higher n-tupling bifurcations ($n > 2$) in the Chirikov standard map by a simple method. His method has some similarity to the two-resonance approximate renormalization of Escande and Doveil[9] used for invariant curves. He reconstitutes approximately a local standard map about a single island of period n ($n > 2$). Through this procedure he obtains a recurrence relation between the old parameter of the original map and the new one of the local map and calculates the accumulation point as the fixed point of the recurrence relation.

The approximate renormalization method used in this paper may be called the method of quadratic approximants. Since the self-similarity is an asymptotic property valid only in the immediate vicinity of a periodic point, it is sufficient to retain up to quadratic terms in the Taylor expansion of the composed maps. Although the linear approximant yields quite a lot of information, it is necessary to make quadratic approximants in order to fully resolve the two scaling factors as will be seen in the later sections. Derrida and Pomeau[10] compared the linear approximants for T, T^2 and T^3, and obtained the accumulation point and the bifurcation ratio for 1/2-and 1/3-bifurcation. The quadratic approximant is formed by keeping the terms to the second order in the Taylor expansion of the n-th iterate of a map, T^n. Comparison of successive approximants of a 1/n-bifurcation sequence gives the accumulation point P*, the bifurcation ratio δ, the scaling factors α and β, and the universal residue value R*. By looking at the recurrence relation between T^{n^l} and $T^{n^{l+1}}$ with large l, we can make better approximations, and obtain the approximate universal map T*:

$$T^* \simeq A^l \, T^n_{p^*} A^{-l}, \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Actually Helleman[11] and Helleman and Mackay[12] used this method to calculate the accumulation point, the bifurcation ratio, and the scaling factors for 1/2-bifurcation. In their calculations they compare quadratic approximants for T and T^2, the lowest pair of approximants. We made better approximations by comparing the next higher approximants, for T^2 and T^4, and obtained P*, δ, α and β for 1/4-bifurcation by comparing the quadratic approximants for T and T^4. Although approximants for low order iterates can be handled analytically, it is imperative to resort to the numerical method for quadratic approximants for high order iterates of T for 1/n-bifurcation with high n values and high order calculations. Thus by the numerical implementation of this simple method, we obtained universal maps, scaling factors, bifurcation ratios and universal residue values for 1/n-bifurcation (n = 2, 3, 4, 5, 6).

Chang et al.[13], and Hu and Mao[14] used similar methods for tricritical behaviors and period-doubling bifurcations in 1-dim maps with 2-parameters and 1-parameter, respectively. However our objective in this paper is different from theirs, and the maps under study are essentially different. We are concerned about an area-preserving map, while they were concerned about a 1-dim dissipative map. Our objective is to study the critical behavior of higher n-tupling bifurcatins (n = 3,4,5,6) in 2-dim area-preserving maps; there are no analogous phenomena in 1-dim maps. Finally, it should be noted that the only renormalization analysis for higher n-tupling bifurcation reported prior to this paper is one by Lichtenberg[8]. However his renormalization scheme is entirely different from ours. In effect, his one-shot renormalization scheme amounts to our lowest l (l = 1) approximation.

Our method can also be applied to other infinitely nested structures such as critical invariant curves[15, 16]. We applied the method to a noble invariant curve and obtained the critical parameter value, the scaling factors and the universal residue value which have been obtained through other methods[9, 16, 17].

In section 2, we describe the renormalization method briefly and present the results for 1/n-bifurcation calculated using this method. In section 3, we discuss the application of the method to a noble invariant curve. In section 4, we present a summary and further discussion of our results.

II. AN APPROXIMATE RENORMALIZATION OF M / N-BIFURCATION

In this paper, we use the DeVogelaere quadratic map, since that map is represented in terms of symmetry coordinates. The DeVogelaere map is

$$T_p : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' = -y + f_p(x) \\ y' = x - f_p(x') \end{pmatrix}, f_p(x) = px - (1-p)x^2.$$

The DeVolgelaere map is an area-preserving map with unit Jacobian ($\det(DT)$=1). Here DT is the Jacobian matrix which is the two by two matrix of partial derivatives of x' and y' with respect to x and y. The map is also reversible since it can be factored into the product $(T_p S)$ S of two involutions:

$$S : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' = x \\ y' = -y \end{pmatrix}$$

$$T_p \cdot S : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' = y + f_p(x) \\ y' = x - f_p(x') \end{pmatrix}$$

Two symmetry lines formed from the points invariant under S and TS are y=0 and $y = x - f_p(x)$. If (x_i, y_i) is an orbit of T_p, then $S(x_i, y_i)$ is an orbit of T_p^{-1} which is the inverse map of T_p, where $T_p^{-1} = S \cdot T_p S$. A symmetric orbit which is its own time reversal is an invariant orbit under S. Thus, a symmetric orbit must have two symmetric points on the symmetry lines. When the period n of a symmetric periodic orbit is even the two symmetric points lie on the symmetry line y=0, and when n is odd one symmetric point is on the symmetry line y=0 and the other symmetric point on the symmetry line $y = x - f_p(x)$[16].

The stability of an periodic orbit is determined by the Jacobian matrix M of T_p^n about the orbit. The residue R of the periodic orbit is:

$$R = (2 - \mathrm{Tr}(M)) / 4$$ [18].

When $R < 0$ the orbit is hyperbolic, when $0 < R < 1$ it is elliptic, and when $R > 1$ it is hyperbolic with reflection. For an elliptic orbit, the residue can be represented as $R = \sin^2(\pi\omega)$. Here ω is the rotation frequency about a point on the elliptic orbit. When ω is a rational number m/n ($\omega \leq 1/2$), m/n-bifurcation occurs. Rimmer[19] analyzed bifurcations of symmetric periodic orbits in reversible area-preserving maps. When $0 < m/n \leq 1/2$, Rimmer has shown that all periodic orbits produced by generic bifurcation of a symmetric periodic orbit are also symmetric. Therefore, at least one symmetric point of the periodic orbits formed by m/n-bifurcations must lie on the symmetry line y=o.

As an example we take 1/2-bifurcation and describe our method briefly. Let us denote the symmetric periodic point of period 2^l as $(\hat{x}_l, 0)$. The idea of Helleman's method is to associate, for each value p′, a value p such that $T_p^{2^l}$ with origin $(\hat{x}_l, 0)$ looks the same as $T_{p'}^{2^{(l-1)}}$ with origin $(\hat{x}_{l-1}, 0)$ on a small spatial scale (to the second order). Therefore,

$$T_{p'}^{2^{(l-1)}} = \Lambda T_p^{2^l} \Lambda^{-1}, \quad \Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}. \tag{1}$$

If we denote $T_p^{2^l}$ as

$$T_p^{2^l} : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' = F_p^{(l)}(x,y) \\ y' = G_p^{(l)}(x,y) \end{pmatrix}, \text{ then}$$

$$F_p^{(l)}(\hat{x}_l, 0) = \hat{x}_l \text{ and } G_p^{(l)}(\hat{x}_l, 0) = 0.$$

Let us first make Taylor expansions of $F_p^{(l)}$ and $G_p^{(l)}$ about $(\hat{x}_l, 0)$. Then,

$$\begin{aligned} F_p^{(l)}(x,y) &= \hat{x}_l + A_l(p)(x-\hat{x}_l) + B_l(p)y \\ &\quad + U_l(p)\cdot(x-\hat{x}_l)^2 + V_l(p)\cdot(x-\hat{x}_l)\cdot y \\ &\quad + W_l(p)y^2 + \cdots\cdots, \\ G_p^{(l)}(x,y) &= C_l(p)(x-\hat{x}_l) + D_l(p)\cdot y \\ &\quad + Q_l(p)\cdot(x-\hat{x}_l)^2 + R_l(p)\cdot(x-\hat{x}_l)\cdot y \\ &\quad + S_l(p)\cdot y^2 + \cdots\cdots. \end{aligned}$$

Because the self-similarity holds in the vicinity of the periodic point, we expect that it would be sufficient to keep only the terms to the second order in the Taylor expansions. Let us define the linearized map $M_p^{(l)}(x, y)$ of $T_p^{2^l}(x, y)$ as

$$M_p^{(l)}(x,y) = DT_p^{2^l}(x,y) = \begin{pmatrix} H_p^{(l)}(x,y) & I_p^{(l)}(x,y) \\ J_p^{(l)}(x,y) & K_p^{(l)}(x,y) \end{pmatrix} \tag{2}$$

by introducing four functions $H_p^{(l)}$, $I_p^{(l)}$, $J_p^{(l)}$ and $K_p^{(l)}$. The area-preserving condition is given by

$$H_p^{(l)} \cdot K_p^{(l)} - I_p^{(l)} \cdot J_p^{(l)} = 1 \tag{3}$$

for any (x,y).

Then, the coefficients of linear terms of Taylor expansions can be represented by

$$\begin{aligned} &A_l(p) = H_p^{(l)}(\hat{x}_l, 0),\ B_l(p) = I_p^{(l)}(\hat{x}_l, 0), \\ &C_l(p) = J_p^{(l)}(x_l, 0), \text{ and } D_l(p) = K_p^{(l)}(\hat{x}_l, 0) \\ &\quad = (1 + B_l \cdot C_l)/A_l. \end{aligned} \tag{4}$$

Using symmetry coordinates[16], we have $A_l(p) = D_l(p)$. Therefore, the trace of the Jacobian matrix of T_p^{2l} about $(\hat{x}_l, 0)$, $TrM_l(p)$, is given by

$$\begin{aligned} TrM_l(p) &= A_l(p) + D_l(p) \\ &= 2A_l(p). \end{aligned} \tag{5}$$

The coefficients of quadratic terms are also represented by the derivatives of the above four functions, i.e.,

$$U_l(p) = \frac{1}{2}\cdot\frac{\partial H_p^{(l)}}{\partial x}\Big|_{(\hat{x}_l, 0)},\quad V_l(p) = \frac{\partial I_p^{(l)}}{\partial x}\Big|_{(\hat{x}_l, 0)},$$

$$Q_l(p) = \frac{1}{2}\cdot\frac{\partial J_p^{(l)}}{\partial x}\Big|_{(\hat{x}_l, 0)},\quad W_l(p) = \frac{1}{2}\cdot\frac{\partial I_p^{(l)}}{\partial y}\Big|_{(\hat{x}_l, 0)},$$

$$R_l(p) = \frac{\partial K_p^{(l)}}{\partial x}\Big|_{(\hat{x}_l, 0)} = (2B_l \cdot Q_l + C_l \cdot V_l - 2D_l \cdot U_l)/D_l, \text{ and}$$

$$S_l(p) = \frac{1}{2}\cdot\frac{\partial K_p^{(l)}}{\partial y}\Big|_{(\hat{x}_l,0)} = (B_l\cdot R_l + 2C_l\cdot W_l - D_l\cdot V_l)/(2\cdot D_l). \quad (6)$$

The area-preserving condition (3) is used to express D_l, R_l and S_l in terms of the other coefficients.

By eq. (1), we have

$$A_{l-1}(p') = A_l(p),\ D_{l-1}(p') = D_l(p),\ B_{l-1}(p') = \frac{\alpha}{\beta}B_l(p),$$

$$C_{l-1}(p') = \frac{\beta}{\alpha}C_l(p),\ U_{l-1}(p') = \frac{1}{\alpha}U_l(p),\ V_{l-1}(p') = \frac{1}{\beta}V_l(p),$$

$$W_{l-1}(p') = \frac{\alpha}{\beta^2}W_l(p),\ Q_{l-1}(p') = \frac{\beta}{\alpha^2}Q_l(p),\ R_{l-1}(p') = \frac{1}{\alpha}R_l(p) \text{ and}$$

$$S_{l-1}(p') = \frac{1}{\beta}S_l(p). \quad (7)$$

Comparing the diagonal coefficients of the linear terms of $T_p^{2^l}$ with origin $(\hat{x}_l, 0)$ and $T_p^{2^{(l-1)}}$ with origin $(\hat{x}_{l-1}, 0)$, we can obtain the accumulation point p* and the bifurcation ratio δ as follows. By eq. (5) and eq. (7), we have

$$TrM_{l-1}(p') = TrM_l(p). \quad (8)$$

This recurrence relation (8) can also be obtained by the method of Derrida and Pomeau[10]. The fixed point of the recurrence relation (8) gives the accumulation point p*, i.e.,

$$TrM_{l-1}(p^*) = TrM_l(p^*), \quad (9)$$

and

$$\delta = \frac{dp'}{dp}\Big|_{p^*} = \frac{dTrM_l(p)}{dp}\Big|_{p^*} \Big/ \frac{dTrM_{l-1}(p')}{dp'}\Big|_{p^*} \quad (10)$$

gives the bifurcation ratio. As l increases, we naturally obtain more accurate values. For the first two orders, explicit analytic recurrence relations can be easily derived, and Derrida and Pomeau[10] obtained p* and δ to the second order. We extend the calculations to higher order. Our method is as follows. For any given (x_1, y_1), the function-values of $H_p^{(l)}$, $I_p^{(l)}$, $J_p^{(l)}$, and $K_p^{(l)}$ are easily calculated from

$$M_p^{(l)}(x_1,y_1) = \begin{pmatrix} H_p^{(l)}(x_1,y_1) & I_p^{(l)}(x_1,y_1) \\ J_p^{(l)}(x_1,y_1) & K_p^{(l)}(x_1,y_1) \end{pmatrix} = \prod_{i=1}^{2} m_i,$$

where $m_i = DT_p = \begin{pmatrix} f_p'(x_i) & -1 \\ 1 - f_p'(x_{i+1})\cdot f_p'(x_i) & f_p'(x_{i+1}) \end{pmatrix}$,

$f_p'(x_i) = x_i - 2\cdot(1-p)x_i$ and

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = T_p\begin{pmatrix} x_i \\ y_i \end{pmatrix}. \quad (11)$$

Therefore, $TrM_l(p)$ is readily calculated after finding the periodic orbit of period 2^l. That is,

$$TrM_l(p) = 2\cdot H_p^{(l)}(\hat{x}, 0). \quad (12)$$

The equation for p to be solved is given by

$$F(p) \equiv TrM_l(p) - TrM_{l-1}(p) = 0. \quad (13)$$

The root of F(p) = 0 is just the accumulation point p*. The universal residue value R* is: R* = $(2-TrM_l(p^*))/4$. The bifurcation ratio δ is the ratio of the slopes of $TrM_l(p)$ and $TrM_{l-1}(p')$ at the accumulation point p*, eq.(10). At p=p*, the derivatives of $TrM_l(p)$ and $TrM_{l-1}(p)$ are calculated by the ordinary numerical differentiation routine. We numerically solve eq. (13) and eq. (10), and obtain p* and δ to the 6th order (see Table 1). Table 1 shows that we have more accurate values as l increases.

Once we have p*, we can obtain the scaling factors α and β through eq. (7). At the accumulation point p*, we first find the symmetric periodic point of period 2^l and the function-values of $H_p^{(l)}$, $I_p^{(l)}$, $J_p^{(l)}$ and $K_p^{(l)}$ at $(\hat{x}_l, 0)$ through eq. (11). Next we calculate their deriva-

Table 1. The various quantities obtained by an approximate renormalization method for 1/2-bifurcation. Known best values are those obtained by Greene et al. (1981) by following the 1/2-sequence.

l	p*	δ	α	β
1	−1.265564	9.0623	−4.1204	17.012
2	−1.266321	8.6845	−4.0059	16.294
3	−1.26631115	8.72541	−4.01992	16.3729
4	−1.2663112786	8.720596	−4.01775	16.3627
5	−1.266311276899	8.721156	−4.01814	16.3641
6	−1.2663112769223	8.721090	−4.01806	16.36386
known best values	−1.2663112769221	8.721097	−4.0180767	16.363897

tive values at $(\hat{x}_l, 0)$ by the ordinary differentiation routine. Therefore, at p = p* we obtain the coefficients of the terms to the second order of $T_{p*}^{2^l}$ and $T_{p*}^{2^{l-1}}$. By eq. (7), for p = p′= p*, we obtain not only the ratio of the scaling factors α/β, comparing the off-diagonal coefficients of the linear terms, but also obtain the scaling factors α and β separately, comparing the quadratic coefficients of $T_{p*}^{2^l}$ and $T_{p*}^{2^{(l-1)}}$. As l increases we obtain more accurate values of α and β as we see in the Table 1.

We define a renormalization transformation τ by

$$T' = \tau(T) = \Lambda T^2 \Lambda^{-1}$$

with Λ as defined in eq. (1). Let us define T_l as the l-times renormalized map. That is, $T_l = \tau^l(T_0) = \Lambda^l T_0^{2^l} \Lambda^{-l}$. If we put $T_0 = T_p*$, then $\lim_{l\to\infty} T_l = T^*$. Here T* is the fixed point called a universal map under the renormalization transformation. Then we have approximately $T^* \simeq \Lambda^l T_{p*}^{2^l} \Lambda^{-l}$ for large l. Therefore we have an approximate universal map T*:

$$T^* \approx \Lambda^6 \cdot T_{p*}^{2^6} \cdot \Lambda^{-6}$$
$$= \begin{pmatrix} x' = -1.27176x - 1.01322y - 2.31933x^2 \\ +.0334822xy + \cdots\cdots \\ y' = -.609305x - 1.27176y + .793849x^2 \\ +5.91963xy + \cdots\cdots \end{pmatrix},$$

$$R^* = 1.13588.$$

A similar renormalization technique can also be applied to 1/3-, 1/4-, 1/5-, and 1/6-bifurcations. However, since 1/3- and 1/5-bifurcation sequences have "period-2" behavior rather than straight geometric convergence, the renormalization transformation $\tau(T)$ now must be

$$T' = \tau(T) = \Lambda T^{n^2} \Lambda^{-1},$$

where n = 3 and 5. Let T_0 be the initial map. We define T_l by

$$T_l = \tau^l(T_0) = \Lambda^l T_0^{n^{2l}} \Lambda^{-l}.$$

For $T_0 = T_p*$, we have

$$\lim_{l\to\infty} \Lambda^l \cdot T_{p*}^{n^{2l}} \Lambda^{-l} = T^*$$

and

$$\lim_{l\to\infty} \Lambda^l \cdot T_{p*}^{n^{(2l+1)}} \Lambda^{-l} = T^{**}.$$

Unlike 1/2-bifurcation ("period-1" behavior), we have two fixed points T* and T**. We calculated to the 3rd order (l=3) p*, δ, α and β for n = 3. The results are listed in the Table 2. Two approximate universal maps T* and T** are given by

Table 2. The various quantities obtained by an approximate renormalization method for m/n-bifurcation. Known best values are those obtained by following the 1/n-sequence in our previous studies (1984, 1985).

m/n bifurcation	l	p*	δ	α	β
1/3	3	−.477013684274045	407.4254	−43.9794	−186.723
known best values		−.477013684274048	407.422	−43.9807	−186.7
1/4	5	−.0689824440291	24.4616	−5.6119	14.2824
known best values		−.0689824440286	24.45	−5.6141	14.269
1/5	2	.177137427506	401.75	−43.34	−76.09
known best values		.177137427510	401.92	−43.27	−75.70
1/6	5	.3362383932	13.83	−8.248	6.302
known best values		.3362383931	13.85	−8.25	6.30

$T^{**} = BT^*B^{-1}$. Thus T* and T** in eq.(14) are two different fixed points which have the same

$$T^* \approx \Lambda^3 T_{p^*}^{3^6} \Lambda^{-3}$$

$$= \begin{pmatrix} x' = -.46742x - .90841y - 1.737x^2 - .23725xy \\ +\cdots\cdots \\ y' = .86032x - .46742y - .53585x^2 + 1.8279xy \\ +\cdots\cdots \end{pmatrix}$$

and

$$T^{**} \approx \Lambda^3 T_{p^*}^{3^7} \Lambda^{-3}$$

$$= \begin{pmatrix} x' = -.46742x + .28035y - 17.773x^2 - 18.059xy \\ +\cdots\cdots \\ y' = -2.7877x - .46742y - 32.037x^2 - 33.728xy \\ +\cdots\cdots \end{pmatrix}$$

$$\text{with } R^* = .73371. \tag{14}$$

Because of marginal eigenperturbation in coordinate changes corresponding to only scale changes, if T* is a fixed point under a renormalization transformation, $T^{**}(= BT^*B^{-1})$ is also a fixed point, where

$$B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ for any nonzero a and b.}$$

But, in eq. (14) there does not exist a B such that eigenvalues.

Using this technique, we have performed the calculations to higher order for 1/4-, 1/5- and 1/6-bifurcations. The results (p*, δ , α and β) are listed in Table 2. For 1/4-bifurcation, this procedure can be performed analytically for l = 1 comparing 1-cycle and renormalized 4-cycle maps (see the appendix). For 1/4- and 1/6-bifurcations, like 1/2-bifurcation, we obtained a single approximate universal map. For 1/4-bifurcation, we have

$$T^* \approx \Lambda^5 \cdot T_{p^*}^{4^5} \Lambda^{-5}$$

$$= \begin{pmatrix} x' = -.03561x - 1.0114y - 1.251x^2 - .3308xy \\ +\cdots\cdots \\ y' = .9874x - .03561y - .1982x^2 + .4157xy \\ +\cdots\cdots \end{pmatrix}$$

$$\text{with } R^* = .51781$$

and

for 1/6-bifurcation, we have

$$T^* \approx \Lambda^5 \cdot T_{p^*}^{6^5} \Lambda^{-5}$$

$$= \begin{pmatrix} x' = .374x - 1.129y - .764x^2 - .623xy \\ +\cdots\cdots \\ y' = .7617x + .374y + .0588x^2 - .0967xy \\ +\cdots\cdots \end{pmatrix}$$

$$\text{with } R^* = .3130.$$

For 1/5-bifurcation, like 1/3-bifurcation, we again have two universal maps T* and T**. They are

$$T^* \approx \Lambda^2 \cdot T_{p^*}^{5^4} \Lambda^{-2} = \begin{pmatrix} x' = .2217x - .46y - 1.4x^2 - .377xy + \cdots \\ y' = 2.07x + .2217y - .215x^2 + .162xy + \cdots \end{pmatrix}$$

and

$$T^{**} \approx \Lambda^2 \cdot T_{p^*}^{5^5} \Lambda^{-2} = \begin{pmatrix} x' = .2217x + .27y - 15.9x^2 - 3.21xy + \cdots \\ y' = -3.52x + .2217y - 41.4x^2 - 18.3xy + \cdots \end{pmatrix}$$

with $R^* = .3892$.

III. RENORMALIZATION ANALYSIS OF A VIBRATIONAL INVARIANT CURVE

Greene[18] suggested a connection between existence of invariant circles and the stability of nearby periodic orbits. His numerical work shows that for the critical parameter value at which the residues of nearby periodic orbits are 0.25009, the invariant curve with rotation number γ^{-1} breaks up ($\gamma = (1+\sqrt{5})/2$) . Escande and Doveil[9] have reproduced Green's result by an approximate renormalization method. Kadanoff and Shenker[15, 17] and Mackay[16] "explain" the scaling laws for a noble invariant curve by the renormalization method.

We apply our renormalization method used in section 2 to the study of a noble invariant curve of rotation number γ^{-2} in the quadratic map studied by Mackay[16]. The quadratic map we consider in this section is given as

$$T_p : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' = -y + f_p(x) \\ y' = x - f_p(x') \end{pmatrix}, \quad f_p(x) = \frac{p}{2} - \frac{x^2}{2} \tag{15}$$

We consider invariant circles with rotation number ω . Any irrational rotation number ω has a unique infinite fraction represented by

$$\omega = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cdots}}} = [m_1, m_2, m_3, \cdots\cdots], \quad 0 < \omega < 1,$$

where

$$m_i \in Z^+, \quad i = 1, 2, \cdots\cdots .$$

The rational approximant r_n of ω is given by

$$r_n = p_n / q_n, \tag{16}$$

where

$$p_n = m_n p_{n-1} + p_{n-2}, \quad p_{-1} = 1, \quad p_0 = 0$$

and

$$q_n = m_n q_{n-1} + q_{n-2}, \quad q_{-1} = 0, \quad q_0 = 1.$$

The vibrational island chain corresponding to the rational approximant $r_n (= p_n/q_n)$ of ω is formed by p_n/q_n-bifurcation of the fixed point of T_p. As n increases, the vibrational island chain approaches closer to the vibrational invariant curve of rotation number ω . Thus the invariant curve of rotation number ω is formed at the parameter value at which the residue R of the fixed point is $\sin^2(\pi\omega)$.

For $\omega = 1/\gamma^2 = [2, (1,)^\infty]$, Mackay[16] has found that in the critical case ($p^* = 2.38216325...$), the nearby vibrational island chains repeat each other asymptotically on smaller scales. The scaling factors across and along the dominant half-line α and β are given by

$$\alpha = -1.4148360$$

and

$$\beta = -3.0668882..$$

The δ_n-sequence converges to a limit value δ (=1.6280), where δ_n is defined by $\delta_n = (a_{n-1} - a_n)/(a_n - a_{n+1})$. Here, a_n is the parameter value at which the periodic orbit of type (p_n, q_n) has some given residue.

Since there is unstable coordinate change corresponding to quadratic shear, it is necessary to choose the scaling coordinate to kill the component in this direction[16]. The scaling coordinate (X, Y) is given by

$$X = x - Sy^2, \quad S = -0.7783661$$

and

$$Y = y.$$

Thus,

$$T_p : \begin{pmatrix} Y \\ X \end{pmatrix} \to \begin{pmatrix} Y' = X+SY^2 - \frac{p}{2} + \frac{1}{2}[-Y + \frac{p}{2} \\ -\frac{1}{2}(X+SY^2)^2]^2 \\ X' = -Y + \frac{p}{2} - \frac{1}{2}(X+SY^2)^2 \\ -SY'^2 \end{pmatrix}.$$

Asymptotically, on the dominant half-line,

$$T_{p*}^{q_{l-1}} \approx \Lambda T_{p*}^{q_l} \Lambda^{-1}, \quad \Lambda = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}.$$

That is, $T_{p*}^{q_l}$ looks the same as $T_{p*}^{q_{l-1}}$ on a small spatial scale.

Let us denote the dominant symmetric point as $(\hat{X}_l, 0)$. The method we are going to use is to associate, for each p′, a value p such that $T_p^{q_l}$ with origin $(\hat{X}_l, 0)$ looks the same as $T_{p'}^{q_{l-1}}$ with origin $(\hat{X}_{l-1}, 0)$ on a small spatial scale (to the second order). Therefore, $T_{p'}^{q_{l-1}} = \Lambda T_p^{q_l} \Lambda^{-1}$. By the same renormalization technique used for m/n-bifurcations, we calculated to the 17th order p*, δ, α and β. The results are listed in Table 3. The approximate universal map T* to this order is given by

$$T^* \approx \Lambda^{17} T_{p*}^{q_{17}} \Lambda^{-17}$$

$$= \begin{pmatrix} X' = .5079X + 4.779Y + .184Y^2 + .1159XY \\ + \cdots\cdots \\ Y' = -.1552X + .5079Y - .0974Y^2 + \cdots\cdots \end{pmatrix}.$$

The calculated residue value is 0.24625 (the expected value R* = .25009). Thus, when the residues of the nearby periodic orbits are 1/4, the vibrational invariant curve of rotation number γ^{-2} is on the edge of disappearance.

IV. SUMMARY AND DISCUSSION

In this section we summarize and discuss our results and the renormalization method used for the analysis. As was reported[2], the self-similarity repeats in 1/n-bifurcation sequences. It is observed that the pattern repeats itself from one bifurcation to the next for even n (n=2,4,6), while for odd n for every other bifurcation (n=3,5). When $\omega = 1/\gamma^2$, Mackay[16] has found that in the critical case the vibrational island chains repeat each other asymptotically on smaller scales.

Our objective is to study the critical behaviors of higher n-tupling bifurcations and a noble invariant curve by a simple approximate renormalization method. The approximate renormalization method we employed in this paper is essentially a generalization of Helleman's original idea which was used for 1/2-bifurcation to the lowest order. Comparison of the quadratic approximants for T^{n^l} and $T^{n^{l+1}}$ yields the accumulation point p*, the scaling factors α and β, the convergence ratio δ and the universal residue value R*. As shown in section 2, as l increases, the higher order approximation gives better values. Since at the accumulation point, in the limit $l \to \infty$, $\Lambda^l T^{n^l} \Lambda^{-l} \to T^*$, we can also obtain an approximate universal map. By the numerical implementation

Table 3. The various quantities obtained by an approximate renormalization method for an invariant curve of the rotation number γ^{-2} ($\gamma = (1+\sqrt{5})/2$). Known best values were obtained by Mackay (1982).

l	p*	δ	β	α
17	2.382158	1.6279	−3.083	−1.4199
known best values	2.382163	1.6280	−3.0669	−1.4148

of this simple method, we obtained p*, α and β, δ, R* and T* for 1/n-bifurcation (n=2,3,4,5,6) and a noble invariant curve. The results agree well with the values obtained by directly following 1/n-bifurcation sequences and through other methods for a noble invariant curve.

The obtained universal residue values R* for 1/n-bifurcation and the islands near the critical noble invariant curve are less than unity, while R* in the period-doubling bifurcation is 1.13588. The fact that R* is less than unity implies that infinitely nested islands exist, and near these islands stochastic orbits have long-time correlations[3]. Consequently islands play important roles in transport phenomena in area-preserving maps.

ACKNOWLEDGEMENTS

We wish to acknowledge the support of the Ministry of Education, Republic of Korea through a grant to the Research Institute for Basic Sciences, Seoul National University.

APPENDIX

In this appendix we analytically calculate δ, α and β for 1/4-bifurcation (l=1) and for 1/2-bifurcation (l=2) by Helleman's scheme. We take for the quadratic map,

$$T_c : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' = -y + f(x) \\ y' = x - f(x') \end{pmatrix}, \quad f(x) = cx + x^2.$$

Thus,

$$x_{t+1} = -y_t + cx_t + x_t^2 \qquad (A.1)$$

and

$$x_t = y_{t+1} + cx_{t+1} + x_{t+1}^2, \qquad (A.2)$$

so that T has a fixed point $(\hat{x}, \hat{y})$ at (0, 0). Note that this map T_c is equivalent to the DeVogelaere quadratic map T_p up to scale changes in x and y with c=p.

1. 1/4-bifurcation (l=1)

The stable 4-cycle is given by

$$\hat{x}_{4i} = -\frac{1}{2} \cdot \{[(1-c)^2 - 1 + 2((1-c)^2-1)^{\frac{1}{2}}]^{\frac{1}{2}} + c\},$$

$$\hat{x}_{4i+1} = -\frac{1}{2} \cdot \{-((1-c)^2-1)^{\frac{1}{2}} + c\} = \hat{x}_{4i+3}$$

$$\hat{x}_{4i+2} = -\frac{1}{2} \cdot \{-[(1-c)^2 - 1 + 2((1-c)^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}} + \},$$

$$\hat{y}_{4i} = 0 = \hat{y}_{4i+2},$$

$$\hat{y}_{4i+1} = -\frac{1}{2}[(1-c)^2 - 1 + 2((1-c)^2-1)^{\frac{1}{2}}]^{\frac{1}{2}} = -\hat{y}_{4i+3},$$

$$a = c + 2\hat{x}_{4i} = -c - 2\hat{x}_{4i+2} \text{ and } b = c + 2\hat{x}_{4i+1}. \qquad (A.3)$$

Now consider small deviations:

$$x_t = \hat{x}_t + u_t$$

and

$$y_t = \hat{y}_t + v_t. \qquad (A.4)$$

Substituing eq.(A.4) into eq.(A.1) and eq.(A.2), we get

$$u_{t+1} = -v_t + (c + 2\hat{x}_t)u_t + u_t^2 \qquad (A.5)$$

and

$$u_t = v_{t+1} + (c + 2\hat{x}_{t+1})u_{t+1} + u_{t+1}^2. \qquad (A.6)$$

Letting t = 4i+3 in eq.(A.5) and t = 4i+2 in eq.(A.6) and adding them, we get

$$u_{4i+4} = -u_{4i+2} + 2bu_{4i+3} + 2u_{4i+3}^2. \qquad (A.7)$$

Letting t = 4i+2 in eq.(A.5) and t = 4i+1 in eq.(A.6) and adding them, we get

$$u_{4i+3} = -u_{4i+1} - 2au_{4i+2} + 2u_{4i+2}^2. \qquad (A.8)$$

Letting t = 4i+1 in eq.(A.5) and t=4i in eq.(A.6) and adding them, we get

$$u_{4i+2} = -u_{4i} + 2bu_{4i+1} + 2u_{4i+1}^2. \qquad (A.9)$$

Letting t = 4i in eq.(A.5), we get

$$u_{4i+1} = -v_{4i} + a u_{4i} + u_{4i}^2 . \qquad (A.10)$$

Substituting eq.(A.10) into eq.(A.9), we get

$$u_{4i+2} = -2b v_{4i} + (-1 + 2ab) u_{4i} + 2(b + a^2) u_{4i}^2 . \qquad (A.11)$$

Substituting eq.(A.11) and eq.(A.10) into eq.(A.8), we get

$$u_{4i+3} = (1 + 4ab) v_{4i} + (a - 4a^2 b) u_{4i} - [1 + 4ab + 4a^3 - 2(1 - 2ab)^2] u_{4i}^2 . \qquad (A.12)$$

Substituting eq.(A.11) and eq.(A.12) into (A.7), we get

$$u_{4i+4} = 4b(1 + 2ab) v_{4i} + (1 - 8a^2 b^2) u_{4i} + 8ab(4a^3 b - 3a^2 + 2ab^2 - 3b) u_{4i}^2 + \text{higher-order terms} . \qquad (A.13)$$

Similarly we get

$$u_{4i} = -4b(1 + 2ab) v_{4i+4} + (1 - 8a^2 b^2) u_{4i+4} + 8ab(4a^3 b - 3a^2 + 2ab^2 - 3b) u^2_{4i+4} + \text{higher-order terms} . \qquad (A.14)$$

Rescaling eq. (A.13) and eq. (A.14) with

$$x_i = \alpha u_{4i}$$

and

$$y_i = \beta v_{4i}$$

we obtain the renormalized map,

$$x_{i+1} = -y_1 + c' x_i + x_i^2 ,$$
$$x_i = y_{i+1} + c' x_{i+1} + x_{i+1}^2 ,$$
$$c' = 1 - 8a^2 b^2 ,$$
$$\alpha = 8ab(4a^3 b - 3a^2 + 2ab^2 - 3b),$$

and

$$\beta = -4\alpha b(1 + 2ab) . \qquad (A.15)$$

Thus,

$$c' = 1 - 8((1-c)^2 - 1)^2 - 16[(1-c)^2 - 1]^{\frac{3}{2}} . \qquad (A.16)$$

The fixed point of eq.(A.16) c^* is

$$c^* = -.070826 \cdots\cdots .$$

The bifurcation ratio δ is given by

$$\delta = dc'/dc \,\Big|_{c^*} = 24.71 \cdots\cdots .$$

The scaling factors α and β are

$$\alpha = -6.6469 \cdots\cdots ,$$

and

$$\beta = 17.633 \cdots\cdots ,$$

2. 1/2-bifurcation(l=2)

The 2-cycle is given by

$$\hat{x}_{2i} = \frac{1}{2}[-(c+1) + \sqrt{(c+1)(c-3)}] ,$$
$$\hat{x}_{2i+1} = \frac{1}{2}[-(c+1) - \sqrt{(c+1)(c-3)}] ,$$
$$\hat{y}_{2i} = 0 = \hat{y}_{2i+1} ,$$
$$A = 2(c + 2\hat{x}_{2i}) \quad \text{and} \quad B = 2(c + 2\hat{x}_{2i+1}) .$$

In a similar way, we get

$$u_{2i+2} = -B u_{2i} + (-1 + \frac{1}{2}AB) u_{2i} + (B + \frac{A^2}{2}) u_{2i}^2 + \text{higher order terms} \qquad (A.17)$$

and

$$u_{2i} = B v_{2i+2} + (-1 + \frac{1}{2}AB) u_{2i+2} + (B + \frac{A^2}{2}) u_{2i+2}^2 + \text{higher order terms} .$$

Rescaling eq. (A.17) with

$$x_i = u_{2i} \quad \text{and} \quad y_i = v_{2i}$$

We obtain T_1:

$$x_{i+1} = -B y_i + (-1 + \frac{1}{2}AB) x_i + (B + \frac{A^2}{2}) x_i^2$$
$$x_i = B y_{i+1} + (-1 + \frac{1}{2}AB) x_{i+1} + (B + \frac{A^2}{2}) x_{i+1}^2 . \qquad (A.18)$$

The 4-cycle is given by

$$\hat{x}_{4i} = -\frac{1}{2}\{[(1-c)^2 - 1 - 2((1-c)^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}} + c\} ,$$
$$\hat{x}_{4i+1} = -\frac{1}{2}[((1-c)^2 - 1)^{\frac{1}{2}} + c] = \hat{x}_{4i+3} ,$$

$$\hat{x}_{4i+2} = -\frac{1}{2}\{-[(1-c)^2-1-2((1-c)^2-1)^{\frac{1}{2}}]^{\frac{1}{2}} + c\}$$

$$\hat{y}_{4i} = 0 = \hat{y}_{4i+2} ,$$

$$\hat{y}_{4i+1} = -\frac{1}{2}[(1-c)^2-1-2((1-c)^2-1)^{\frac{1}{2}}]^{\frac{1}{2}} = -\hat{y}_{4i+3} ,$$

$$a = c+2\hat{x}_{4i} = -c-2\hat{x}_{4i} \quad \text{and} \quad b = c+2\hat{x}_{4i+1} .$$

In the same way, we get eq.(A.13) and eq.(A.14). Rescaling eq.(A.13) and eq.(A.14), with $X_i = u_{4i}$ and $Y_i = v_{4i}$ we obtain T_2:

$$X_{i+1} = 4b(1+2ab)Y_i + (1-8a^2b^2)X_i + 8ab(4a^3b-3a^2+2ab^2-3b)X_i^2 + \text{higher order terms} \qquad (A.19)$$

and

$$Y_{i+1} = -4b(1+2ab)Y_{i+1} + (1-8a^2b^2)X_{i+1} + 8ab(4a^3b-3a^2+2ab^2-3b)X_{i+1}^2 + \text{higher order terms} .$$

For $l=2$, $T_1 = AT_2A^{-1}$, where

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} .$$

Thus, rescaling eq.(A.19) with

$x_i = \alpha X_i$ and $y_i = \beta Y_i$

we get $T_2(=AT_2A^{-1})$:

$$x_{i+1} = 4b(1+2ab)\frac{\alpha}{\beta}y_i + (1-8a^2b^2)x_i + 8ab(4a^3b-3a^2+2ab^2-3b)/\alpha \cdot x_i^2 \qquad (A.20)$$

and

$$y_{i+1} = -4b(1+2ab)\cdot\frac{\alpha}{\beta}y_{i+1} + (1-8a^2b^2)x_{i+1} + 8ab(4a^3b-3a^2+2ab^2-3b)/\alpha \cdot x^2_{i+1} .$$

Comparing eq.(A.18) and eq.(A.20), we get

$$-1+\frac{1}{2}AB = 1-8a^2b^2 , \qquad (A.20a)$$

$$4b(1+2ab)\cdot\frac{\alpha}{\beta} = -B , \qquad (A.20b)$$

and

$$8ab(4a^3b-3a^2+2ab^2-3b)/\alpha = B+\frac{A^2}{2} . \qquad (A.20c)$$

By eq.(A.20a), we get c^* and δ :

$c^* = -1.266321$

and

$\delta = 8.6845$.

By eq.(A.20b) and eq.(A.20c), we get α and β :

$\alpha = -4.0059$

and

$\beta = 16.294$.

REFERENCES

[1] V.I. Arnold and A. Avez, *Ergodic Problems in Classical Dynamical Systems* (Benjamin, New York, 1968).

[2] K.C. Lee, S.Y. Kim, D.I. Choi, Phys. Lett. **103A**, 225(1984) and J. Korean Phys. Soc. **18**, 243 (1985).

[3] J.D. Meiss and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985).

[4] M.J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978) .

[5] P. Collet, J.P. Eckman and H. Koch, Physica **3D**, 457 (1981).

[6] M. Widom and L.P. Kadanoff, Physica **3D**, 457 (1982).

[7] J.M. Greene, R.S. Mackay, F. Vivaldi and M.J. Feigenbau, Physica **3D**, 468 (1981).

[8] A.J. Lichtenberg, Physica **14D**, 387 (1985).

[9] D.F. Escande and F. Doveil, J. Stat. Phys. **26**, 257 (1981).

10] B. Derrida and Y. Pomeau, Phys. Lett. **80A**, 217 (1980).

[11] R.H.g. Helleman, *Fundamental Problems in Statistical Mechanics* Vol.5, edited by E.D.G. Cohen (North-Holland, Amsterdam, 1980).

[12] R.H.G. Helleman and R.G. Mackay, *Long-time Prediction in Dynamics*, edited by C.W. Horton,

L.E. Reichl and V.G. Szebehely (Wiley, New York, 1983).
[13] S.J. Chang, M. Wortis and J.A. Wright, Phys. Rev. **24A**, 2669 (1981).
[14] B. Hu and J.M. Mao, Phys. Rev. **25A**, 1196 (1982).
[15] S.J. Shenker and L.P. Kadanoff, J. Stat. Phys. **27**, 631 (1982).
[16] R.S. Mackay, Ph.D thesis (Princeton University, 1982).
[17] L.P. Kadanoff, Phys. Rev. Lett. **47**, 1641 (1981).
[18] J.M. Greene, J. Math. Phys. **20**, 1183 (1979).
[19] R. Rimmer, J. Diff. Eqs. **29**, 329 (1978).

면적이 보존되는 본뜨기에서의 m/n갈림과 불변 곡선의 재규격화군 방법의 적용

김 상 윤 · 이 구 철

서울대학교 물리학과

최 덕 인

한국과학기술원 물리학과

(1986 년 7 월 31 일 받음)

2 차원 면적이 보존되는 본뜨기에서 m/n－갈림 연쇄를 재규격화군 방법을 적용하여 분석하였다. 본뜨기 T의 P－돌림을 테일러 전개하여 제 2 차 항까지 취하는 2차형 어림셈 방법을 사용하였다. 이 방법으로 계산된 축적인자는 갈림 연쇄를 쫓아가며 계산된 값과 잘 일치한다. 이 방법은 황금비 불변 곡선의 임계현상 분석에도 적용될 수 있음을 보였다.