Scaling Behaviors of n-tupling Bifurcations with High n in Area-preserving Maps

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There are in general n-tupling bifuractions in area-preserving maps[1,2]. We have studied the period trebling and quadrupling sequences and found unique universal limiting behaviors in area-preserving maps[3]. In the case of the period trebling sequence, the self-similarity is repeated every other bifurcation. On the other hand the period quadrupling sequence repeats the self-similarity every time bifurcation takes place. It would be interesting to see if these behaviors can be seen in n-tupling bifurcation sequences with higher n. Furthermore it would also be interesting to see if there are certain limiting behaviors in the universal numbers such as δ, α and β in a sequence of n, n being the n-tupling bifurcation.

In this work we study period 5-tupling and 6-tupling bifurcations in 2-dimensional reversible area-preserving maps. As shown elsewhere[4], all the quadratic maps are equivalent and we take the following form of the Henon quadratic map of $T: x_{n+1}=-y_n+2h(x_n)$, $y_{n+1}=x_n$ with $h(x)=\frac{1}{2}(1-ax^2)$. The two symmetry lines formed from the invariant points of the orientation-reversing involutions of TS and S, where $S: x_{n+1}=y_n$, $y_{n+1}=x_n$ and $TS: x_{n+1}=2h(y_n)-x_n$, $y_{n+1}=y_n$ are y=x for S and x=h(y) for TS. Since the daughter orbits of symmetric periodic orbits which are formed by 5-tupling and 6-tupling bifurcations also are symmetric periodic orbits, the symmetry lines play important roles in locating the periodic orbits. When the residue R, defined as $R=(2-\mathrm{Tr}M)/4$ with M being the Jacobian matrix of T^{ℓ} about an orbit of period ℓ, is $\sin^2(\pi/5)$ a pair of stable and unstable orbits of period 5ℓ are formed[1]. When one periodic point $p_n(0)$ of the odd period ℓ is on the symmetry line of S, the $(\ell+1)/2$-th point from that point is on the other symmetry line of TS ($P_n(S)=T^{S}P_n(0)$, where $P_n(0)$ is the initial point of the $5^n(=\ell)$-periodic orbit). Shown in Fig. 1 are consecutively enlarged figures of 5-tupling bifurcations associated with two periodic points on S and TS. The center circles inside the pentagon are the lower order periodic points surrounded by the five stable points of the next higher order period 5-tupled orbit. At the parameter value where the $\ell=5^n$, 5^{n+1} and 5^{n+2} orbits are stable, the consecutive pentagons in Fig. 1 are enlarged figures near the region around the previous lower order periodic orbits P(0) (A_n's in the Fig. 1) and $P(\frac{\ell+1}{2})$ (A_n''s in the Fig. 1) on S and TS. As

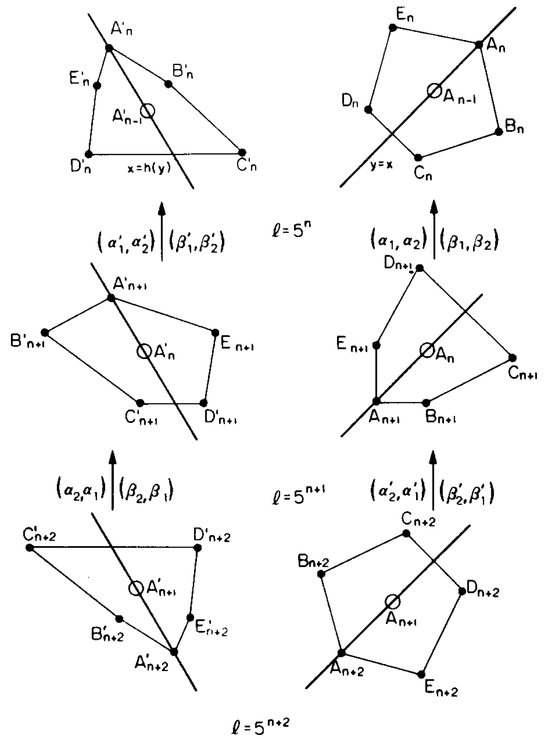

Fig. 1. Period 5-tupling bifurcations associated with two periodic points on S and TS. A_n, B_n, C_n, D_n, E_n, A_n', B_n', C_n', D_n' and E_n' are elements of an ℓ-periodic orbit corresponding to $P_n(0)$, $P_n(\frac{\ell}{5})$, $P_n(\frac{2\ell}{5})$, $P_n(\frac{3\ell}{5})$, $P_n(\frac{4\ell}{5})$, $P_n(\frac{\ell+1}{2})$, $P_n(\frac{\ell+1}{2}+\frac{\ell}{5})$, $P_n(\frac{\ell+1}{2}+\frac{2\ell}{5})$, $P_n(\frac{\ell+1}{2}+\frac{3\ell}{5})$ and $P_n(\frac{\ell+1}{2}+\frac{4\ell}{5})$ of the text with $\ell=5^n$ respectively.

indicated in the figure by (α_1, α_2) and (α_1', α_2'), the higher order periodic point on the symmetry line appears either on the opposite side with respect to the center circle $((\alpha_1, \alpha_2))$ or on the same side $((\alpha_1', \alpha_2'))$, thus flipping the figure $((\alpha_1, \alpha_2))$ or leaving the figure the same way $((\alpha_1', \alpha_2'))$. However this flipping $((\alpha_1, \alpha_2))$ and leaving $((\alpha_1', \alpha_2'))$ occur at the same time, one on one symmetry line and the other on the other symmetry line. In the next enlarged figures (α_1, α_2) and (α_1', α_2') are interchanged and $\alpha_1 \cdot \alpha_2' = \alpha_2 \cdot \alpha_1'$. Therefore the rescaling can be done at every other 5-tupling and the rescaling factor along the symmetry line α is given by $\alpha = \alpha_1 \cdot \alpha_2' = \alpha_2 \cdot \alpha_1'$. Similarly the scaling factor across the symmetry line β is given by $\beta=\beta_1 \cdot \beta_2' = \beta_2 \cdot \beta_1'$. The (α_j, β_j) and (α_j', β_j') are the limiting values of $(\alpha_n(j), \beta_n(j), \alpha_n'(j), \beta_n'(j); j=1, 2)$ (Table 1) when $n \to \infty$. δ is the limit value of the δ_n-sequence, where $\delta_n=(a_{n-1}-a_n)/(a_n-a_{n+1})$. The values of a_n are those at which the residue R of the stable orbit of $\ell=5^n$ period is $\sin^2(\pi/5)$. We obtain the accumulation point $a_\infty = -0.3228971868 \cdots$. In Table 1, the numerical values of 5-tupling sequences are given. We conclude from these numerical values that we find $\alpha_2=-30.118$, $\alpha_1=-6.082$, $\alpha_1'=1.4366$, $\alpha_2'=7.144$, $\beta_2=8.241$, $\beta_1=3.891$, $\beta_1'=-9.186$, $\beta_2'=-19.45$ and $\delta=20.048$. Thus we obtain $\alpha=\alpha_1 \cdot \alpha_2' = \alpha_2 \cdot \alpha_1' = -43.27$, $\beta=\beta_1 \cdot \beta_2' = \beta_2 \cdot \beta_1' = -75.70$.

For the 6-tupling bifurcation, a pair of stable stable and unstable orbits with 6-tupled period are formed when $R=\frac{1}{4}$[1]. Fig. 2 shows that among the higher order periodic points surrounding the lower order points on TS line of $x=h(y)$ either two or none are on the symmetry line. In the "two points on the line" case, $P_n(0)$ and $P_n(\ell/2)$ (A_n and D_n of Fig. 2) surround the center circle of $P_{n-1}(0)$, and the "none on the line" case occurs when all points are in the region surrounding the center circle $P_{n-1}(\frac{\ell}{2})$. In the next enlargement of the figure, two points $P_{n+1}(0)$ and $P_{n+1}(\frac{\ell}{2})$ on the TS line surround $P_n(0)$ while all six points (A'_{n+1}, B'_{n+1}, C'_{n+1}, D'_{n+1}, E'_{n+1} and F'_{n+1} of Fig. 2) around $P_n(\frac{\ell}{2})$ are off the TS line. If we define $\alpha_n(j)$, $\beta_n(j)$ for $j=1, 2, 3, 4, 5$, (Table 2) in order to retain the pattern of the periodic orbits, we find from Table 2 that all $\alpha_n(j)$ and $\beta_n(j)$ approach the same limits irrespective of j, contrary to the case of 5-tupling bifurcations. The rescaling factor along the symmetry line α is -8.25 and the rescaling factor across the symmetry line β is 6.30 where α and β are the limit values of the α_n- and β_n- sequences. Also the

Table 1. Period 5-tupling sequences.

$\alpha_n(1) = \overline{(A_n-B_n)}_s/\overline{(A_{n+1}-B_{n+1})}_s$,

$\alpha_n(2) = \overline{(B_n-C_n)}_s/\overline{(B_{n+1}-C_{n+1})}_s$,

$\alpha'_n(1) = \overline{(A'_n-B'_n)}_y/\overline{(A'_{n+1}-B'_{n+1})}_y$,

$\alpha'_n(2) = \overline{(B'_n-C'_n)}_y/\overline{(B'_{n+1}-C'_{n+1})}_y$,

$\beta_n(1) = \overline{B_n-E_n}/\overline{B_{n+1}-E_{n+1}}$,

$\beta_n(2) = \overline{C_n-D_n}/\overline{C_{n+1}-D_{n+1}}$,

$\beta'_n(1) = \overline{B'_n-E'_n}/\overline{B'_{n+1}-E'_{n+1}}$ and

$\beta'_n(2) = \overline{C'_n-D'_n}/\overline{C'_{n+1}-D'_{n+1}}$

are defined with respect to Fig. 1.

$\overline{(A_n-B_n)}_s$ and $\overline{(B_n-C_n)}_s$ denote respectively the length of the projection of the line $\overline{A_n-B_n}$ and $\overline{B_n-C_n}$ onto the S-symmetry line. $\overline{(A'_n-B'_n)}_y$ and $\overline{(B'_n-C'_n)}_y$ denote respectively the length of the projection of the line $\overline{A'_n-B'_n}$ and the line $\overline{B'_n-C'_n}$ onto the y-axis. $\overline{B_n-E_n}$, $\overline{C_n-D_n}$, $\overline{B'_n-E'_n}$ and $\overline{C'_n-D'_n}$ denote respectively the length of the line $\overline{B_n-E_n}$, line $\overline{C_n-D_n}$, the line $\overline{B'_n-E'_n}$ and the line $\overline{C'_n-D'_n}$.

n	δ_n	$\alpha_n(1)$	$\alpha_n(2)$	$\alpha'_n(1)$	$\alpha'_n(2)$
2	19.9690	−28.4896	−5.63052	7.37095	0.87571
3	20.0877	1.45033	7.15241	−6.29401	−30.0816
4	20.0436	−30.1268	−6.08996	7.10850	1.43947
5	20.0479	1.43664	7.11347	−6.07992	−30.1176
6	20.0476	−30.1178	−6.08221	7.11414	1.43658
7	20.0478	1.43664	7.11391	−6.08250	−30.1183

n	$\beta_n(1)$	$\beta_n(2)$	$\beta'_n(1)$	$\beta'_n(2)$
2	8.69941	4.37160	−17.7107	−8.87985
3	−9.15754	−19.3058	3.76468	8.08063
4	8.23195	3.88462	−19.4840	−9.18802
5	−9.18557	−19.4554	3.89303	8.24277
6	8.24126	3.89141	−19.4532	−9.18571
7	−9.18602	−19.4547	3.89119	8.24111

same definition as before of $\delta_n=(a_{n-1}-a_n)/(a_n-a_{n+1})$ gives the limiting value of $\delta=13.8$ and the accumulation point a_∞ is −0.55942053...

In summary, we find numerically that 5-tupling and 6-tupling sequences have universal limiting behaviors. The intervals in the parameter between successive 5-tupling and 6-tupling bifurcations tend to a geometric progression with a ratio of $1/\delta$ and the pattern of periodic orbits repeats itself asymptotically, for every other 5-tupling bifurcation and from one 6-tupling bifurcation to the next. However it would be difficult to

Table 2. Period 6-tupling sequences.

$\alpha_n(1) = \overline{(A_n - B_n)_y} / \overline{(A_{n+1} - B_{n+1})_y}$,
$\alpha_n(2) = \overline{(B_n - C_n)_y} / \overline{(B_{n+1} - C_{n+1})_y}$,
$\alpha_n(3) = \overline{(C_n - D_n)_y} / \overline{(C_{n+1} - D_{n+1})_y}$,
$\alpha_n(4) = \overline{(A'_n - B'_n)_y} / \overline{(A'_{n+1} - B'_{n+1})_y}$,
$\alpha_n(5) = \overline{(B'_n - C'_n)_y} / \overline{(B'_{n+1} - C'_{n+1})_y}$

$\beta_n(1) = \overline{B_n - F_n} / \overline{B_{n+1} - F_{n+1}}$,
$\beta_n(2) = \overline{C_n - E_n} / \overline{C_{n+1} - E_{n+1}}$,
$\beta_n(3) = \overline{A'_n - F'_n} / \overline{A'_{n+1} - F'_{n+1}}$,
$\beta_n(4) = \overline{B'_n - E'_n} / \overline{B'_{n+1} - E'_{n+1}}$,
$\beta_n(5) = \overline{C'_n - D'_n} / \overline{C'_{n+1} - D'_{n+1}}$

are defined with respect to Fig. 2.

$\overline{(A_n - B_n)_y}$, $\overline{(B_n - C_n)_y}$, $\overline{(C_n - D_n)_y}$, $\overline{(A'_n - B'_n)_y}$ and $\overline{(B'_n - C'_n)_y}$ denote respectively the length of the projection onto the y-axis of the line $\overline{A_n - B_n}$, the line $\overline{B_n - C_n}$, the line $\overline{C_n - D_n}$, the line $\overline{A'_n - B'_n}$ and the line $\overline{B'_n - C'_n}$. $\overline{B_n - F_n}$, $\overline{C_n - E_n}$, $\overline{A' - F'_n}$, $\overline{B'_n - E'_n}$ and $\overline{C'_n - D'_n}$ denote respectively the length of the line $\overline{B_n - F_n}$, the line $\overline{C_n - E_n}$, the line $\overline{A'_n - F'_n}$, the line $\overline{B'_n - E'_n}$ and the line $\overline{C'_n - D'_n}$.

n	δ_n	$\alpha_n(1)$	$\alpha_n(2)$	$\alpha_n(3)$	$\alpha_n(4)$	$\alpha_n(5)$
2	13.907	−9.7553	−9.0846	−9.2121	0.73225	−5.2934
3	13.819	−8.0180	−8.2082	−8.2658	−10.322	−10.0357
4	13.835	−8.3363	−8.2841	−8.2598	−7.8561	−8.0944
5	13.846	−8.2485	−8.2572	−8.2615	−8.2810	−8.2824
6	13.852	−8.2573	−8.2542	−8.2525	−8.2326	−8.2543

n	$\beta_n(1)$	$\beta_n(2)$	$\beta_n(3)$	$\beta_n(4)$	$\beta_n(5)$
2	6.0640	5.5918	12.205	11.984	11.661
3	6.2894	6.3528	5.2244	5.2003	5.1288
4	6.2895	6.2815	6.4537	6.4525	6.4568
5	6.2999	6.3004	6.2857	6.2838	6.2805
6	6.3033	6.3030	6.3064	6.3057	6.3047

conclude that there is any tendency to approach a limit in, say, the sequence of δ's as we see in the table 3.

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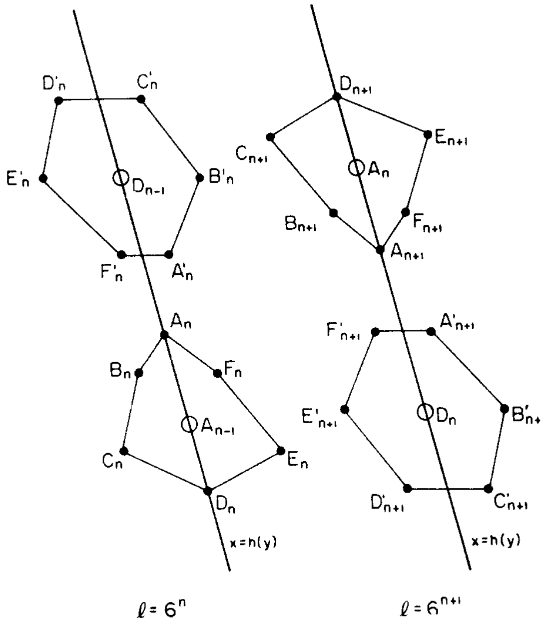

Fig. 2. Period 6-tupling bifurcations associated with two periodic points on S. A_n, B_n, C_n, D_n, E_n, F_n, A_n', B_n',C_n', D_n', E_n', F_n' are elements of an ℓ-periodic orbit corresponding to $P_n(0)$, $P_n(\frac{\ell}{6})$, $P_n(\frac{2\ell}{6})$, $P_n(\frac{3\ell}{6})$, $P_n(\frac{4\ell}{6})$, $P_n(\frac{5\ell}{6})$, $P_n(\frac{3\ell}{36})$, $P_n(\frac{9\ell}{36})$, $P_n(\frac{15\ell}{36})$,$P_n(\frac{21\ell}{36})$ and $P_n(\frac{27\ell}{36})$ of the text with $\ell=6^n$ respectively.

Table 3. Three universal constants δ, α and β of n-tupling bifurcation sequences with n from 2 to 6. Constants with n=2 to 3 are taken from refs. 3 and 4. α and β with odd n are the geometric mean of 2 scaling constants, i.e. $\bar{\alpha}=\sqrt{\alpha_1\alpha_2}$, $\bar{\beta}=\sqrt{\beta_1\beta_2}$.

n	δ	α	β
2	8.721	−4.018	16.36
3	20.2	6.63	13.67
4	24.5	−5.61	14.3
5	20.05	6.58	8.70
6	13.85	−8.25	6.30

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면적이 보존되는 본뜨기에서 n이 클때 n－곱 갈림에서의 축척 거동

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면적이 보존되는 2차원 가역 본뜨기에서 5곱과 6곱 갈림 현상을 연구하였다. 5곱과 6곱 갈림현상에도 축척 거동에 보편적 극한이 있음을 보였고 이 극한값들을 계산하였다.