Critical Behavior of Period n-Tuplings in Coupled Maps

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We study the critical behavior of period n-tuplings ($n=2,3,4,\ldots$) in two coupled one-dimensional (1D) maps. Using a renormalization method, the critical behavior associated with coupling is particularly investigated in the zero-coupling case in which the two 1D maps become uncoupled. It is found that the zero-coupling fixed map of the period n-tupling renormalization transformation has two relevant "coupling eigenvalues" (CE's) associated with coupling perturbations, α and n (α is the orbital scaling factor of 1D maps). In the linear-coupling case, in which the coupling function has a leading linear term, the scaling associated with coupling is governed by two CE's, α and n, whereas it is governed by only one CE, n, in the nonlinear-coupling case in which the leading term is nonlinear.

Universal scaling behavior of period n-tuplings ($n = 2,3,4,\ldots$) has been found in a one-parameter family $f_A(x)$ of one-dimensional (1D) unimodal maps with quadratic maxima. As the parameter A increases, an initially stable orbit loses its stability and gives birth to a stable period-doubled orbit. An infinite sequence of such period-doubling bifurcations accumulates at a finite parameter value A_∞ and exhibits a universal asymptotic scaling behavior. [1,2]

What happens beyond the period-doubling accumulation point A_∞ is interesting from the viewpoint of chaos. The parameter interval between A_∞ and the final boundary-crisis point A_c beyond which no periodic or chaotic attractors can be found within the unimodality interval is called the "chaotic" regime. Within this region, the parameter values with chaotic attractors form a set of positive measure. [3] These "chaotic" parameter values are found in between an infinite number of windows with stable periodic attractors. Besides the period-doubling sequence (the $n = 2$ case), higher period n-tupling ($n = 3,4,\ldots$) sequences of periodic orbits with periods n^k ($k = 1,2,\ldots$) can be selected from the infinitely many periodic windows densely embedded in the chaotic regime. Unlike the period-doubling sequence, stability regions of periodic orbits in the higher period n-tupling sequences are not adjacent on the parameter axis because they are born by their own tangent bifurcations. The asymptotic scaling behaviors of these (disconnected) higher period n-tupling sequences characterized by the orbital and parameter scaling factors, α and δ, vary depending on n. [2,4–11]

In this paper, we study the critical behavior of period n-tuplings ($n = 2,3,4,\ldots$) in a map T consisting of two identical 1D maps coupled symmetrically:

$$T: \begin{cases} x_{i+1} = F(x_i, y_i) = f(x_i) + g(x_i, y_i), \\ y_{i+1} = F(y_i, x_i) = f(y_i) + g(y_i, x_i), \end{cases} \tag{1}$$

where the subscript i denotes the discrete time, $f(x)$ is a 1D unimodal map with a quadratic maximum at $x = 0$, and $g(x,y)$ is a coupling function. The uncoupled 1D map f satisfies the normalization condition $f(0) = 1$, and the coupling function g obeys the condition $g(x,x) = 0$ for any x. This coupled map may help us to understand how coupled nonlinear oscillators, such as Josepson-junction arrays or chemically reacting cells, exhibit various dynamical behaviors. [12–14]

The period-doubling case ($n = 2$) was previously studied in Refs. [15–20]. Here, we extend the results for the $n = 2$ case to all the other higher period n-tupling cases (i.e., the cases of $n = 3,4,\ldots$) in the zero-coupling case where the two 1D maps become uncoupled. In particular, the critical behavior associated with coupling is investigated by the renormalization method developed in Refs. [15] and [19].

The period n-tupling ($n = 2,3,\ldots$) renormalization transformation $\mathcal{N}$ for a coupled map T is composed of the n-times iterating ($T^{(n)}$) and rescaling (B) operators:

$$\mathcal{N}(T) \equiv BT^{(n)}B^{-1}. \tag{2}$$

Here, the rescaling operator B is

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \tag{3}$$

because we consider only in-phase orbits ($x_i = y_i$ for all i).

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Applying the renormalization operator $\mathcal{N}$ to the coupled map (1) k times, we obtain the k-times renormalized map T_k of the form

$$T_k : \begin{cases} x_{i+1} = F_k(x_i, y_i) = f_k(x_i) + g_k(x_i, y_i), \\ y_{i+1} = F_k(y_i, x_i) = f_k(y_i) + g_k(y_i, x_i). \end{cases} \quad (4)$$

Here, f_k and g_k are the uncoupled and coupling parts of the k -times renormalized function F_k, respectively. They satisfy the following recurrence equations:

$$f_{k+1}(x) = \alpha f_k^{(n)}(\frac{x}{\alpha}), \quad (5)$$

$$g_{k+1}(x,y) = \alpha F_k^{(n)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - \alpha f_k^{(n)}(\frac{x}{\alpha}), \quad (6)$$

where $F_k^{(n)}(x,y) = F_k(F_k^{(n-1)}(x,y), F_k^{(n-1)}(y,x))$.

The recurrence relations (5) and (6) define a renormalization operator $\mathcal{R}$ transforming a pair of functions (f, g):

$$\begin{pmatrix} f_{k+1} \\ g_{k+1} \end{pmatrix} = \mathcal{R} \begin{pmatrix} f_k \\ g_k \end{pmatrix}. \quad (7)$$

The renormalization transformation $\mathcal{R}$ obviously has a fixed point (f^*, g^*) with $g^*(x,y) = 0$, which satisfies $\mathcal{R}(f^*, 0) = (f^*, 0)$. Here f^* is just the 1D fixed function satisfying

$$f^*(x) = \alpha f^{*(n)}(\frac{x}{\alpha}) \quad (8)$$

where $\alpha = 1/f^{*(n-1)}(1)$ due to the normalization condition $f^*(0) = 1$. The fixed point $(f^*, 0)$ governs the critical behavior near the zero-coupling critical point because the coupling fixed function is identically zero, i.e., $g^*(x,y) = 0$. Here, we restrict our attention to this zero-coupling case.

Consider an infinitesimal coupling perturbation $(0, \varphi)$ to the zero-coupling fixed point $(f^*, 0)$. We then examine the evolution of a pair of functions (f^*, φ) under $\mathcal{R}$. Linearizing $\mathcal{R}$ at the zero-coupling fixed point, we obtain a linearized "coupling operator" $\mathcal{L}_c$ transforming a coupling perturbation φ:

$$\begin{aligned} \varphi_{k+1}(x,y) &= [\mathcal{L}_c \varphi_k](x,y) && (9) \\ &= \alpha\, \delta[F_k^{(n)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_k^{(n)}(\frac{x}{\alpha})] \equiv \alpha\, [F_k^{(n)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_k^{(n)}(\frac{x}{\alpha})]_{\rm linear} && (10) \\ &= \alpha f^{*\prime}(f^{*(n-1)}(\frac{x}{\alpha}))\, \delta[F_k^{(n-1)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_k^{(n-1)}(\frac{x}{\alpha})] \\ &\quad + \alpha \varphi_k(f^{*(n-1)}(\frac{x}{\alpha}), f^{*(n-1)}(\frac{y}{\alpha})). && (11) \end{aligned}$$

Here, the prime denotes a derivative, and the variation $\delta[F_k^{(n)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_k^{(n)}(\frac{x}{\alpha})]$ is introduced as the linear term (denoted by $[F_k^{(n)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_k^{(n)}(\frac{x}{\alpha})]_{\rm linear}$) in φ for the deviation of $F_k^{(n)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_k^{(n)}(\frac{x}{\alpha})$ from 0. If a coupling perturbation $\varphi^*(x)$ satisfies

$$\nu\, \varphi^*(x,y) = [\mathcal{L}_c \varphi^*](x,y), \quad (12)$$

then it is called a coupling eigenperturbation with eigenvalue ν.

However, it is not easy to directly solve the coupling-eigenvalue equation (12). We, therefore, introduce a tractable recurrence equation for a "reduced coupling eigenfunction" of $\varphi^*(x,y)$ [15,19] defined by

$$\Phi^*(x) \equiv \left. \frac{\partial \varphi^*(x,y)}{\partial y} \right|_{y=x} \quad (13)$$

Differentiating Eq. (12) with respect to y and setting $y = x$, we obtain an eigenvalue equation for a reduced linearized coupling operator $\tilde{\mathcal{L}}_c$:

$$\begin{aligned} \nu\, \Phi^*(x) &= [\tilde{\mathcal{L}}_c \Phi^*](x) && (14) \\ &= \delta F_2^{(n)}(\frac{x}{\alpha}) = [F_2^{(n)}(\frac{x}{\alpha})]_{\rm linear} && (15) \\ &= f^{*\prime}(f^{*(n-1)}(\frac{x}{\alpha}))\, \delta F_2^{(n-1)}(\frac{x}{\alpha}) \\ &\quad + f^{*(n-1)\prime}(\frac{x}{\alpha}) \Phi^*(f^{*(n-1)}(\frac{x}{\alpha})). && (16) \end{aligned}$$

Here, $F(x,y) = f^*(x) + \varphi^*(x,y)$, $F_2^{(n)}(x)$ is a "reduced function" of $F^{(n)}(x,y)$ defined by $F_2^{(n)}(x) \equiv \partial F^{(n)}(x,y)/\partial y|_{y=x}$, and the variation $\delta F_2^{(n)}(\frac{x}{\alpha})$ is also introduced as the linear term (denoted by $[F_2^{(n)}(\frac{x}{\alpha})]_{\rm linear}$) in Φ^* of the deviation of $F_2^{(n)}(\frac{x}{\alpha})$ from 0.

In the case $n = 2$, the variation $\delta F_2^{(2)}(\frac{x}{\alpha})$ of Eq. (15) becomes

$$\delta F_2^{(2)}(\frac{x}{\alpha}) = \Phi^*(\frac{x}{\alpha}) f^{*\prime}(f^*(\frac{x}{\alpha})) + f^{*\prime}(\frac{x}{\alpha}) \Phi^*(f^*(\frac{x}{\alpha})). \quad (17)$$

Substituting $\delta F_2^{(2)}(\frac{x}{\alpha})$ into Eq. (16), we have $\delta F_2^{(3)}(\frac{x}{\alpha})$ for $n = 3$, which consists of three terms,

$$\begin{aligned} \delta F_2^{(3)}(\frac{x}{\alpha}) &= \Phi^*(\frac{x}{\alpha}) f^{*\prime}(f^*(\frac{x}{\alpha})) f^{*\prime}(f^{*(2)}(\frac{x}{\alpha})) \\ &\quad + f^{*\prime}(\frac{x}{\alpha}) \Phi^*(f^*(\frac{x}{\alpha})) f^{*\prime}(f^{*(2)}(\frac{x}{\alpha})) \\ &\quad + f^{*\prime}(\frac{x}{\alpha}) f^{*\prime}(f^*(\frac{x}{\alpha})) \Phi^*(f^{*(2)}(\frac{x}{\alpha})). && (18) \end{aligned}$$

Repeating this procedure successively, we obtain $\delta F_2^{(n)}(\frac{x}{\alpha})$ for a general n, composed of n terms,

$$\delta F_2^{(n)}(\frac{x}{\alpha}) = \sum_{i=0}^{n-1} f^{*(i)\prime}(\frac{x}{\alpha}) \Phi^*(f^{*(i)}(\frac{x}{\alpha})) \times f^{*(n-i-1)\prime}(f^{*(i+1)}(\frac{x}{\alpha})) \quad (19)$$

where $f^{(0)}(x) = x$.

Using the fact that $f^{*\prime}(0) = 0$, it can be easily shown that when $x = 0$, the reduced coupling eigenvalue equation (16) becomes

$$\nu\,\Phi^*(0) = [\prod_{i=1}^{n-1} f^{*\prime}(f^{*(i)}(0))]\,\Phi^*(0) = \alpha\Phi^*(0). \tag{20}$$

There are two cases. If the coupling eigenfunction $\varphi^*(x,y)$ has a leading linear term, its reduced coupling eigenfunction $\Phi^*(x)$ becomes nonzero at $x = 0$. In this case of $\Phi^*(0) \neq 0$, we obtain the first CE

$$\nu_1 = \alpha. \tag{21}$$

The eigenfunction $\Phi_1^*(x)$ with CE ν_1 has the form

$$\Phi_1^*(x) = 1 + a_1^* x + a_2^* x^2 + \cdots. \tag{22}$$

In the other case of $\Phi^*(0) = 0$, we find that $f^{*\prime}(x)$ is an eigenfunction for the reduced CE equation (16). Since Eq. (19) for the case $\Phi^*(x) = f^{*\prime}(x)$ becomes

$$\delta F_2^{(n)}(\frac{x}{\alpha}) = n f^{*(n)\prime}(\frac{x}{\alpha}), \tag{23}$$

the reduced CE equation reduces to

$$\nu f^{*\prime}(x) = n f^{*\prime}(x). \tag{24}$$

Hence, we obtain the second relevant CE

$$\nu_2 = n \tag{25}$$

with reduced coupling eigenfunction $\Phi_2^*(x) = f^{*\prime}(x)$. It is also found that there exists an infinite number of additional (coordinate change) reduced eigenfunctions $f^{*\prime}(x)\,[f^{*l}(x) - x^l]$ with irrelevant CE's α^{-l} $(l = 1, 2, \ldots)$, which are associated with coordinate changes. We conjecture that together with the two (noncoordinate change) relevant CE's ($\nu_1 = \alpha$, $\nu_2 = n$), they give the whole spectrum of the reduced linearized coupling operator $\tilde{\mathcal{L}}_c$ of Eq. (14) and the spectrum is complete.

In order to see the effect of the CE's on the stability multipliers of the periodic orbits in the period n-tupling sequences, we consider an infinitesimal coupling perturbation $g(x,y) = \varepsilon\varphi(x,y)$ to a critical map at the zero-coupling critical point, in which case the two-coupled map is of the form

$$T : \begin{cases} x_{i+1} = F(x_i, y_i) = f_{A_\infty^{(n)}}(x_i) + g(x_i, y_i), \\ y_{i+1} = F(y_i, x_i), \end{cases} \tag{26}$$

where $A_\infty^{(n)}$ denotes the accumulation value of the parameter A for the period n-tupling case, and ε is an infinitesimal coupling parameter. The map T at $\varepsilon = 0$ is just the zero-coupling critical map consisting of two uncoupled 1D critical maps. It is attracted to the zero-coupling fixed map consisting of two uncoupled 1D fixed maps under iterations of the period n-tupling renormalization transformation $\mathcal{N}$ of Eq. (2).

The reduced coupling function $G(x)$ of $g(x,y)$ is given by [see Eq. (13)]

$$G(x) = \varepsilon\Phi(x) \equiv \varepsilon \left.\frac{\partial\varphi(x,y)}{\partial y}\right|_{y=x}. \tag{27}$$

The kth image Φ_k of Φ under the reduced linearized coupling operator $\tilde{\mathcal{L}}_c$ of Eq. (14) is of the form

$$\begin{aligned} \Phi_k(x) &= [\tilde{\mathcal{L}}_c^k \Phi](x) \\ &\simeq \alpha_1 \nu_1^k \Phi_1^*(x) + \alpha_2 \nu_2^k f^{*\prime}(x) \text{ for large } k \end{aligned} \tag{28}$$

because the irrelevant part of Φ_k becomes negligibly small for large k. Here, α_1 and α_2 are some constants.

The stability multipliers $\lambda_{1,k}$ and $\lambda_{2,k}$ of the n^k-periodic orbit of the map T of Eq. (26) are the same as those of the fixed point of the k-times renormalized map $\mathcal{N}^k(T)$, [19] which are given by

$$\lambda_{1,k} = f_k'(\hat{x}_k), \quad \lambda_{2,k} = f_k'(\hat{x}_k) - 2G_k(\hat{x}_k). \tag{29}$$

Here, f_k is the uncoupled part of the kth image of $(f_{A_\infty^{(n)}}, g)$ under the renormalization transformation $\mathcal{R}$, $G_k(x)$ is the reduced coupling function of the coupling part $g_k(x,y)$ of the kth image, and $\hat{x}_k$ is just the fixed point of $f_k(x)$ [i.e., $\hat{x}_k = f_k(\hat{x}_k)$] and converges to the fixed point x^* of the 1D fixed map $f^*(x)$ as $k \to \infty$. In the critical case ($\varepsilon = 0$), $\lambda_{2,k}$ is equal to $\lambda_{1,k}$, and they converge to the 1D critical stability multiplier $\lambda^* = f^{*\prime}(x^*)$, the value of which varies depending on n. Since $G_k(x) \simeq [\tilde{\mathcal{L}}_c^k G](x) = \varepsilon\Phi_k(x)$ for infinitesimally small ε, $\lambda_{2,k}$ has the form

$$\begin{aligned} \lambda_{2,k} &\simeq \lambda_{1,k} - 2\varepsilon\Phi_k \\ &\simeq \lambda^* + \varepsilon\left[e_1\nu_1^k + e_2\nu_2^k\right] \text{ for large } k \end{aligned} \tag{30}$$

where $e_1 = -2\alpha_1\Phi_1^*(x^*)$ and $e_2 = -2\alpha_2 f^{*\prime}(x^*)$. Hence, the slope S_k of $\lambda_{2,k}$ at the zero-coupling point ($\varepsilon = 0$) is

$$S_k \equiv \left.\frac{\partial\lambda_{2,k}}{\partial\varepsilon}\right|_{\varepsilon=0} \simeq e_1\nu_1^k + e_2\nu_2^k \text{ for large } k. \tag{31}$$

Here, the coefficients e_1 and e_2 depend on the initial reduced function $\Phi(x)$ because the constants α_1 and α_2 are determined only by $\Phi(x)$. Note that the magnitude of the slope S_k increases with k unless both e_1 and e_2 are zero.

We choose monomials x^l $(l = 0, 1, 2, \ldots)$ as the initial reduced functions $\Phi(x)$ because any smooth function $\Phi(x)$ can be represented as a linear combination of monomials by a Taylor series. Expressing $\Phi(x) = x^l$ as a linear combination of eigenfunctions of $\tilde{\mathcal{L}}_c$, we have

$$\begin{aligned} \Phi(x) = x^l &= \alpha_1\Phi_1^*(x) + \alpha_2 f^{*\prime}(x) \\ &\quad + \sum_{l=1}^{\infty} \beta_l f^{*\prime}(x)[f^{*l}(x) - x^l] \end{aligned} \tag{32}$$

where α_1 is nonzero only for $l = 0$, and hence zero for $l \geq 1$, and all β_l's are irrelevant components. Therefore, the slope S_k for large k becomes

$$S_k \simeq \begin{cases} e_1\alpha^k + e_2 n^k & \text{for } l = 0, \\ e_2 n^k & \text{for } l \geq 1. \end{cases} \quad (33)$$

There are two kinds of coupling. In the case of a linear coupling, in which the coupling function $\varphi(x,y)$ has a leading linear term, the reduced coupling function $\Phi(x)$ has a leading constant term. However, for any other nonlinear-coupling case, in which the coupling function has a leading nonlinear term, the reduced coupling function contains no constant term. It, therefore, follows from Eq. (33) that the growth of S_k for large k is governed by the two relevant CEs $\nu_1 = \alpha$ and $\nu_2 = n$ for the linear-coupling case ($l = 0$), but by only the second relevant CE $\nu_2 = n$ for the other nonlinear-coupling cases ($l \geq 1$).

As an example, we numerically study the period-tripling case ($n = 3$) in the two-coupled 1D maps (26) with $f(x) = 1 - Ax^2$ and $\varphi(x,y) = \frac{1}{m}(y^m - x^m)$ ($m = 1, 2, \ldots$), and we confirm the renormalization results (33). For this period-tripling case, we follow the periodic orbits of period 3^k up to level $k = 9$ and obtain the slopes S_k of Eq. (31) at the zero-coupling critical point $(A_\infty, 0)$ ($A_\infty = 1.786\,440\,255\,563\,639\,354\,534\,447\ldots$) when the reduced coupling function $\Phi(x)$ is a monomial x^l ($l = 0, 1, \ldots$).

The sequence of slopes $\{S_k\}$ for the linear-coupling case with $l = 0$ obeys well a two-term scaling law, [20,21]

$$S_k = d_1 r_1^k + d_2 r_2^k, \quad \text{for large } k, \quad (34)$$

where d_1 and d_2 are some constants, $r_1 = -9.277\,341\cdots$, and $r_2 = 2.999\cdots$. Note that the numerical values of r_1 and r_2 agree well with the two relevant CE's $\nu_1 = \alpha$ ($= -9.277\,341\cdots$) and $\nu_2 = 3$. However, in all the other nonlinear-coupling cases ($l = 1, 2, 3$) studied, the sequences of slopes $\{S_k\}$ obey well a one-term scaling law,

$$S_n = d_1 r_1^n \quad (35)$$

where d_1 is some constant and $r_1 = 2.999\,999\,999\cdots$. The value of r_1 is very close to the second CE $\nu_2 = 3$. An extended version of this work including a detailed account of the numerical results, the results for many-coupled cases, and so on will be given elsewhere.[22]

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