Barriers to Transport in Area-Preserving Maps of Class-C^2 and C^0

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Two area preserving maps of class-C^2 and C^0 are numerically studied to investigate a possible dependence of transport barriers on the analyticity class. In the case of a map of class-C^2, a noble invariant curve persists below the critical parameter value and the critical behaviors are the same as the cases of analytic perturbations. However in a map of class-C^0, the rational invariant curves revive and play the role of transport barriers of the stochastic orbit.

I. INTRODUCTION

Let us consider a periodically time-dependent 1-D system governed by a Hamiltonian H;

$$H = H_0(I) - \varepsilon \cdot V(\theta) \cdot \sum_{l=-\infty}^{\infty} \delta(t-l)$$

$$= H_0(I) - \varepsilon \sum_{l=-\infty}^{\infty} \sum_{m=1}^{\infty} V_m \cos 2\pi(m\theta - lt),$$

$$V(\theta) = V(-\theta), \quad V(\theta) = V(\theta - 1)$$

$$\text{and} \quad \int_0^1 V(\theta)\, d\theta = 0$$

The perturbation represents a 'kick' per unit time. It will be convenient to limit our consideration to the cases with $d^2H_0/dI^2 > 0$. This restriction leads us to a twist map.

By constructing a surface of section at $t=0$ (mod 1) in the (I,θ,t)- space, a Poincare time-1 map T can be obtained:

$$T : \begin{pmatrix} I_n \\ \theta_n \end{pmatrix} \to \begin{pmatrix} I_{n+1} = I_n - \varepsilon F(\theta_n) \\ \theta_{n+1} = \theta_n + \omega(I_{n+1}) \end{pmatrix};$$

$$F(\theta) = V'(\theta) \quad \text{and} \quad \omega = H_0'(I).$$

First, under a sufficiently small perturbation, we consider the phase flows near r/s-resonance. The resonant terms in H, those with $l/m = r/s$, dominate the phase flows near r/s-resonance. The largest term is that with the smallest l and m, since Fourier coefficients tend to decrease as m increases. Therefore, the phase flows in the neighborhood of r/s-resonance are dominated by the resonance Hamiltonian H_R: $H_R = H_0(I) + \varepsilon \cdot V_s \cos 2\pi(s\theta - rt)$. By introducing a slow time τ, one can see the phase flows near one island in the r/s-island chain, where $\tau = t/s$. To be the canonical transformation, H_R must be transformed to $\bar{H}_R$, where $H_R = s^{-1}\bar{H}_R$. Transformation to new canonical variables p and Ψ by the generating function

$$S(p, \theta) = s \cdot (\theta - r \cdot \tau) p$$

yields

$$\overline{H}_R = s H_0(ps) + \varepsilon s V_s \cos 2\pi(\Psi - s r \tau)$$

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The equilibrium points $(P = \hat{I}/s, \hat{\Psi})$ of $\bar{H}_R$ are determined by the equation,

$$H_0'(\hat{I}) \pm \varepsilon V_s'(\hat{I}) = r/s,$$

where plus or minus sign corresponds to the case with $\hat{\Psi} = 0$ or $1/2$. Taylor-expansion of $\bar{H}_R$ about the equilibrium point and keeping the terms to the 2nd order, yields

$$\hat{H}_R = \frac{1}{2} s^3 H_0''(\hat{I}) P^2 - \varepsilon \cdot s \cdot V_s(\hat{I}) \cos Q;$$

where $P = p - \hat{p}$ and $Q = 2\pi\Psi$. Phase flows near the equilibrium point are dominated by $\bar{H}_R$. This approximation is called the pendulum approximation.

Linearizing around the equilibrium point yields

$$\dot{\xi} = s^3 H_0''(\hat{I}) \cdot \eta, \quad \dot{\eta} = \varepsilon s V_s(\hat{I}) \cdot \cos \dot{Q} \cdot \xi,$$

where the dot denotes the derivative with respect to τ, $\xi = Q - \hat{Q}$, $\eta = P$ and $\hat{Q} = 0$ or π. The solution for the time displacement operator $M(\tau)$ satisfying

$$\begin{pmatrix} \xi(\tau) \\ \eta(\tau) \end{pmatrix} = M(\tau) \begin{pmatrix} \xi(0) \\ \eta(0) \end{pmatrix}$$

is

$$M(\tau) = \begin{pmatrix} \cos \hat{\omega}\tau & (s^3 \cdot H_0''(\hat{I})/\hat{\omega}) \cdot \sin \hat{\omega}\tau \\ -(\hat{\omega}/s^3 H_0''(\hat{I})) \cdot \sin \hat{\omega}\tau & \cos \hat{\omega}\tau \end{pmatrix};$$

$\hat{\omega}^2(r/s) = -\varepsilon s^4 \cdot V_s(\hat{I}) \cdot H_0''(\hat{I}) \cdot \cos \hat{Q}$ Since $\hat{Q}$ is 0 or π, $\hat{\omega}$ is a real or pure imaginary number. The residue $R(r/s)$ of the periodic orbit in the Poincare time-1 map is defined[2] by

$$R(r/s) = [2 - TrM(1)]/4 = \sin^2\left(\frac{\hat{\omega}}{2}\right).$$

For $0 < \hat{\omega} < \pi$, the periodic orbit is stable. If the perturbation $V(\theta)$ is a function of class $-C^n$,

$$V_s \sim \frac{1}{s^{n+2}} \text{ and } \hat{\omega}^2(r/s) \sim \frac{\varepsilon}{s^{n-2}} \qquad (1)$$

Greene connected the existence of an invariant curve to the stability of nearby periodic orbits[2]. Greene's residue criterion indicates that, if the residues of nearby orbits go to zero asymptotically, then an invariant curve seems to exist. By equation (1), under a sufficiently small perturbation of class- C^n $(n > 2)$, $\hat{\omega}^2(r/s) \to 0$ and $R(r/s) \to 0$ as $s \to \infty$ for any r/s-resonance. Therefore, by the pendulum approximation and Green's residue criterion, the critical smoothness of the perturbation n_c is two for the Hamiltonian and one for the map. This agrees with the result of Chirikov obtained by the pendulum approximation and the resonance overlap criterion.[1,3]

By Moser's twist theorem, the sufficient critical smoothness of perturbation for a map T is three [4]. Therefore, for a map T with perturbation of class-C^2, the persistence of invariant curves is not guaranteed by Moser's twist theorem, but by the above estimates. In this paper, we study whether or not a noble invariant curve persists for a map T with $F(\theta)$ of class-C^2. By Greene's residue criterion, we show numerically that the noble invariant curves exist below the critical perturbation strength and the critical behaviors seem to be the same as those for analytic perturbations.

For a map with $F(\theta)$ of class-C^0, since the smoothness of the perturbation is below the critical smoothness, all invariant curves are expected to be broken and extended chaos occurs for arbitrarily small perturbations. But Chirikov observed that at a finite ε extended chaos did not occur, and thus the chaotic region was confined in I[1,3]. This curious phenomenon was first observed during numerical studies by Hine (referred to by Chirikov[1,3]). We also observed the same phenomena for certain parameter intervals and then a complete barrier to transport seemed to exist. We show that rational invariant curves revive and turn into complete barriers to transport at certain parameter values. Succession of revival of rational invariant curves in a specific parameter interval makes the rational invariant curves play the role of an effective barrier in that interval.

In section 2, we show numerically the existence of a noble invariant curve and describe the critical behaviors for a map T with $F(\theta)$ of class-C^2. In section 3 we show, analytically and numerically, the revival of rational invariant curves for a map T with $F(\theta)$ of class-C^0. In the final section, we summarize and discuss our results.

II. EXISTENCE AND CRITICAL BEHAVIORS OF A NOBLE INVARIANT CURVE IN A MAP OF CLASS-C^2

We study an area-preserving map of class-C^2, T which has a unit Jacobian ($\det(DT)=1$),

$$T: \begin{cases} I' = I - \varepsilon F(\theta) \\ \theta' = \theta - I \end{cases}$$

where

$$F(\theta) = \begin{cases} 4\theta^3 - 3/4\cdot\theta, & 0 \le \theta \le 1/4 \\ -4(\theta-1/2)^3 - 3/4\cdot(\theta-1/2), & 1/4 \le \theta \le 3/4 \\ 4(\theta-1)^3 - 3/4\cdot(\theta-1), & 3/4 \le \theta \le 1. \end{cases}$$

$$F(\theta) = F(\theta+1) \quad \text{and} \quad \int_0^1 F(\theta)\, d\theta = 0.$$

Here DT is the Jacobian matrix of T which is the two by two matrix of partial derivatives of θ' and I' with respect to θ and I. Since $T(\theta+1, I) = (\theta'+1, I')$, T is a periodic map in θ (angle variable) and can be represented on a cylinder. An orbit of T has a winding number ω if the average number of rotations per iteration of T ($\lim_{n\to\infty} (\theta_n - \theta_0)/n$) exists, where $(\theta_n, I_n) = T^n(\theta_0, I_0)$.

Since T has a rotational shear ($\partial\theta'/\partial I|_\theta > 0$), T is a twist map and can be obtained from a generating function $L(\theta, \theta')$ such that $I = -\partial L(\theta, \theta')/\partial\theta$ and $I' = \partial L(\theta, \theta')/\partial\theta'$. Where [illegible] $(\theta-\theta')^2 - V(\theta)$ and $F(\theta)$ [illegible] tionary action principle, a sequence [illegible] $\theta_{t+q} = \theta_t + p$ yields a periodic orbit [illegible] number p/q if its action A [illegible] stationary with respect to variation [illegible] $= \theta_0 + p$.

Since T has a reversible symmetry [illegible] factored into the product $(TS)\cdot S$ of [illegible] tions, where

$$S: \begin{cases} \theta' = -\theta + n \\ I' = I + \varepsilon F(\theta) \end{cases} \quad ; n \in Z \text{ (integer)}$$

The four symmetry half-lines formed [illegible] invariant points under S and $T\cdot S$ are [illegible] $I = 2\theta$ and $I = 2\theta - 1$. If $\{(\theta_i, I_i)\}$ is an orbit [illegible] then $\{S(\theta_i, I_i)\}$ is an orbit of T^{-1} when [illegible] TS) is the inverse map of T. A symmetric [illegible] is an orbit which is its own time [illegible] fore symmetric orbits are invariant [illegible] must have two points on the symmetry [illegible] A reversible, periodic, area-preserving [illegible] has two symmetric periodic orbits for each [illegible] al winding number ω in the range [illegible] One of these orbits minimizes the [illegible] the other minimaximizes the action.

The residue R of an orbit of period q [illegible] by $R = (2 - \lambda - \lambda^{-1})$, where λ and λ^{-1} are [illegible] values of the Jacobian matrix of T^q [illegible] orbit[2]. When $R<0$ the orbit [illegible] when $0<R<1$ it is elliptic, and when [illegible] hyperbolic with reflection. [illegible] minimizing periodic orbits have [illegible] and minimaximizing periodic [illegible] residues[6].

Any irrational winding number [illegible] infinite continued fraction [illegible]

$$\omega = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cdots}} \quad [illegible]$$

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where $m_0 \in Z$ and $m_i \in Z^+, i = 1, 2, \cdots\cdots$. The rational approximant r_n of ω is given by

$$r_n = p_n / q_n,$$

$$p_{n+1} = m_{n+1} p_n + p_{n-1}, \; p_{-2} = 0 \text{ and } p_{-1} = 1$$

and

$$q_{n+1} = m_{n+1} q_n + q_{n-1}, \; q_{-2} = 1 \text{ and } q_{-1} = 0.$$

The difference in actions F_n between minimizing and minimaximizing of winding number p_n/q_n can be interpreted as the area which is transported between minimizing and minimaximizing periodic orbits per iteration[7,8]. Mather has shown that given a sequence of rationals $p_n/q_n \to \omega$, there exists an invariant circle of winding number ω if and only if $F_\omega = \lim_{n\to\infty} F_n = 0$[9]. When F_ω is nonzero, Mather, Aubry and Katok has shown that a hyperbolic invariant set of rotation number ω which is called a cantorus exists.[5,9,10]. Cantori can be regarded as circles with an infinity of gaps caused by the overlap of nearby resonances.

We study whether or not an invariant curve of winding number γ^{-2} exists under a perturbation $(\gamma = (1+\sqrt{5})/2)$. Following Mackay, we find the parameter values ε_n such that the minimaximizing symmetric periodic orbit of winding number p_n/q_n corresponding to the nth rational approximant to γ^{-2} has some given residue, e.g. 1[11]. The limit value ε^* of $\{\varepsilon_n\}$-sequence is the critical parameter value for the invariant curve. The convergence ratio δ is the limit value of $\{\delta_n\}$-sequence defined by $\delta_n = (\varepsilon_{n-1} - \varepsilon_n)/(\varepsilon_n - \varepsilon_{n+1})$. These sequences are shown in the TABLE 1. By super-converging the results, we obtained ε^* and δ:

$$\varepsilon^* = 1.3630577 \text{ and } \delta = 1.6280.$$

Therefore, the parameter value ε_n accumulates geometrically at the critical value ε^*:

$$\varepsilon_n - \varepsilon^* \sim \delta^{-n}$$

Table 1. Parameter value ε_n for the nth convergent minimaximizing periodic orbit to γ^{-2} to have residue 1 and the corresponding convergence ratio; $\gamma = (1+\sqrt{5})/2$.

n	ε_n	δ_n
3	1.72791753	
4	1.56942565	1.94941
5	1.48812316	1.59105
6	1.43702338	1.78782
7	1.40844114	1.58694
8	1.39043019	1.71734
9	1.37994250	1.59142
10	1.37335234	1.66866
11	1.36940298	1.60707
12	1.36694548	1.64267
13	1.36544944	1.61811
14	1.36452488	1.63471
15	1.36395930	1.62591
16	1.36361144	1.62855
17	1.36339784	1.62802
18	1.36326664	1.62793
19	1.36318605	1.62815
20	1.36313655	1.62791
21	1.36310614	

For the subcritical, critical and supercritical cases, we calculated the residues R_n^- and R_n^+ of the minimizing and minimaximizing periodic orbits of winding number p_n/q_n, their action difference F_n and the θ-coordinate θ_n of the nearest minimizing periodic point to the dominant symmetry line ($\theta = 0$). All the minimaximizing periodic orbits tend to have a point on the dominant symmetry $(\theta = 0)$. This observation is not yet understood mathmathically. The other three symmetry lines are called subdominant half-lines. The results are tabulated in the tables (from TABLE II to TABLE VIII).

Table II. Residue value R_n^- for the nth convergent minimaximizing periodic orbit to γ^{-2} when $\varepsilon = \varepsilon^* - \Delta\varepsilon$, ε^*, $\varepsilon^* + \Delta\varepsilon$: $\gamma = (1+\sqrt{5})/2$. $\Delta\varepsilon = 10^{-3}$

n		$\varepsilon^* - \Delta\varepsilon$	ε^*	$\varepsilon^* + \Delta\varepsilon$
3		2.4275×10^{-1}	.24383	2.4490×10^{-1}
4		2.5281×10^{-1}	.25468	2.5656×10^{-1}
5		2.4077×10^{-1}	.24370	2.4667×10^{-1}
6		2.4931×10^{-1}	.25409	2.5896×10^{-1}
7		2.3767×10^{-1}	.24531	2.5319×10^{-1}
8		2.3974×10^{-1}	.25216	2.6523×10^{-1}
9	R_n^-	2.2855×10^{-1}	.24819	2.6953×10^{-1}
10		2.1946×10^{-1}	.25084	2.8675×10^{-1}
11		2.0079×10^{-1}	.24949	3.1015×10^{-1}
12		1.7606×10^{-1}	.25059	3.5711×10^{-1}
13		1.4061×10^{-1}	.24996	4.4539×10^{-1}
14		9.7913×10^{-2}	.25002	6.4207×10^{-1}
15		5.4693×10^{-2}	.25009	1.1646
16		2.1251×10^{-2}	.25010	3.0983
17		4.5081×10^{-3}	.25008	1.5553×10^{-1}
18		4.2434×10^{-4}	.25005	2.2475×10^{-2}
19		3.3628×10^{-5}	.25005	1.8266×10^{-4}
20		7.1529×10^{-6}	.24999	2.3227×10^{-7}

Table III Residue value R_n^- for the nth convergent minimizing periodic orbit to γ^{-2} when $\varepsilon = \varepsilon^* - \Delta\varepsilon$, ε^*, $\varepsilon^* + \Delta\varepsilon$: $\gamma = (1+\sqrt{5})/2$. $\Delta\varepsilon = 10^{-3}$.

n		$\varepsilon^* - \Delta\varepsilon$	ε^*	$\varepsilon^* + \Delta\varepsilon$
3		-2.4742×10^{-1}	−.24854	-2.4965×10^{-1}
4		-2.5107×10^{-1}	−.25301	-2.5496×10^{-1}
5		-2.3960×10^{-1}	−.24263	-2.4570×10^{-1}
6		-2.5674×10^{-1}	−.26173	-2.6682×10^{-1}
7		-2.4286×10^{-1}	−.25078	-2.5897×10^{-1}
8		-2.4444×10^{-1}	−.25737	-2.7099×10^{-1}
9		-2.3254×10^{-1}	−.25296	-2.7521×10^{-1}
10	R_n^-	-2.2321×10^{-1}	−.25577	-2.9321×10^{-1}
11		-2.0410×10^{-1}	−.25455	-3.1787×10^{-1}
12		-1.7914×10^{-1}	−.25631	-3.6793×10^{-1}
13		-1.4235×10^{-1}	−.25556	-4.6206×10^{-1}
14		-9.8819×10^{-2}	−.25529	-6.7336×10^{-1}
15		-2.1332×10^{-2}	−.25545	−3.5658
17		-4.5270×10^{-3}	−.25544	-2.0119×10^{-1}
18		-4.2689×10^{-4}	−.25537	-3.1750×10^{2}
19		-2.7293×10^{-5}	−.25539	-2.6190×10^{4}
20		-1.1374×10^{-5}	−.25533	-1.6148×10^{7}

Table Ⅳ. Action difference F_n between minimaximizing and minimizing periodic orbits nth convergent to γ^{-2} when $\varepsilon = \varepsilon^* - \Delta\varepsilon$, ε^*, $\varepsilon^* + \Delta\varepsilon$; $\gamma = (1+\sqrt{5})/2$, $\Delta\varepsilon = 10^{-3}$.

n		$\varepsilon^* - \Delta\varepsilon$	ε^*	$\varepsilon^* + \Delta\varepsilon$
3		3.5759×10^{-4}	3.5910×10^{-4}	3.6061×10^{-4}
4		8.7155×10^{-5}	8.7771×10^{-5}	8.8391×10^{-5}
5		1.9044×10^{-5}	1.9267×10^{-5}	1.9492×10^{-5}
6		4.6220×10^{-6}	4.7054×10^{-6}	4.7902×10^{-6}
7		1.0143×10^{-6}	1.0451×10^{-6}	1.0768×10^{-6}
8		2.3514×10^{-7}	2.4664×10^{-7}	2.5868×10^{-7}
9		5.1822×10^{-8}	5.6032×10^{-8}	6.0572×10^{-8}
10	F_n	1.1489×10^{-8}	1.3037×10^{-8}	1.4788×10^{-8}
11		2.4374×10^{-9}	2.9936×10^{-9}	3.6737×10^{-9}
12		4.9606×10^{-10}	6.9317×10^{-10}	9.6625×10^{-10}
13		9.2165×10^{-11}	1.5936×10^{-10}	2.7351×10^{-10}
14		1.4998×10^{-11}	3.6692×10^{-11}	8.8038×10^{-11}
15		1.9669×10^{-12}	8.4643×10^{-12}	3.4528×10^{-11}
16		1.8020×10^{-13}	1.9504×10^{-12}	1.8067×10^{-11}
17		9.0380×10^{-13}	4.4945×10^{-13}	1.3281×10^{-11}
18		2.0163×10^{-16}	1.0356×10^{-13}	1.2377×10^{-11}
19		3.3880×10^{-18}	2.3868×10^{-14}	1.2316×10^{-11}
20		2.2246×10^{-19}	5.5006×10^{-15}	

Table Ⅴ. Ratio of the fluxes F_n/F_{n+1} for the critical case.

n	F_n/F_{n+1}
3	4.0913
4	4.5555
5	4.0946
6	4.5023
7	4.2374
8	4.4018
9	4.2980
10	4.3548
11	4.3188
12	4.3496
13	4.3433
14	4.3349
15	4.3397
16	4.3396
17	4.3400
18	4.3390
19	4.3391

When $\varepsilon < \varepsilon^*$ the residues $R_n^\pm$ approach to zero, and when $\varepsilon > \varepsilon^*$, $R_n^\pm$ diverge to $\pm\infty$. For the critical case, $R_n^\pm$ approach to some finite values $R_n^\pm$:

$$R_+^* = 0.250$$

and

$$R_-^* = -0.255$$

Therefore, when the residues of nearby periodic minimaximizing orbits are nearly 1/4, the invariant curve of winding number γ^{-2} is on the edge of disappearance.

When $\varepsilon = \varepsilon^*$, the ratio of the fluxes F_n/F_{n+1} approaches to some value ξ. Here ξ is the area-scaling factor of nearby minimaximizing islands and the observed value of ξ is 4.3390. Therefore, F_n obeys a power law decay: $F_n \sim q_n^{-d_0}$, $d_0 = \log_\gamma \xi$. For the subcritical case, approaches to zero at a rate faster than that for the critical

Table VI. θ-coordinate θ_n of the nth convergent minimizing periodic point nearest to the dominant symmetry line when $\varepsilon = \varepsilon^* - \Delta\varepsilon$, ε^*, $\varepsilon^* + \Delta\varepsilon$; $\Delta\varepsilon = 10^{-3}$.

n		$\varepsilon^* - \Delta\varepsilon$	ε^*	$\varepsilon^* + \Delta\varepsilon$
3		1.5271×10^{-1}	1.5277×10^{-1}	1.5284×10^{-1}
4		1.0956×10^{-1}	1.0965×10^{-1}	1.0973×10^{-1}
5		7.6817×10^{-2}	7.6921×10^{-2}	7.7025×10^{-2}
6		5.4429×10^{-2}	5.4555×10^{-2}	5.4682×10^{-2}
7		3.8269×10^{-2}	3.8417×10^{-2}	3.8566×10^{-2}
8		2.7049×10^{-2}	2.7221×10^{-2}	2.7395×10^{-2}
9		1.8993×10^{-2}	1.9192×10^{-2}	1.9395×10^{-2}
10	θ_n	1.3353×10^{-2}	1.3580×10^{-2}	1.3817×10^{-2}
11		9.3276×10^{-3}	9.5852×10^{-3}	9.8617×10^{-3}
12		6.4878×10^{-3}	6.7778×10^{-3}	7.1033×10^{-3}
13		4.4[illegible]67 $\times 10^{-3}$	4.7885×10^{-3}	5.1768×10^{-3}
14		3.0354×10^{-3}	3.3856×10^{-3}	3.8602×10^{-3}
15		2.0235×10^{-3}	2.3926×10^{-3}	2.9934×10^{-3}
16		1.3191×10^{-3}	1.6911×10^{-3}	2.4869×10^{-3}
17		8.4100×10^{-4}	1.1953×10^{-3}	2.2715×10^{-3}
18		5.2738×10^{-4}	8.4480×10^{-4}	2.2258×10^{-3}
19		3.2790×10^{-4}	5.9709×10^{-4}	2.2227×10^{-3}
20		2.0318×10^{-4}	4.2201×10^{-4}	

Table VII. Exponent x_0^n when $\varepsilon = \varepsilon - \Delta\varepsilon$, ε^*, $\varepsilon^* + \Delta\varepsilon$; $x_0^n = \ln(\theta_n/\theta_{n-1})/\ln\gamma$, $\gamma = (1+\sqrt{5})/2$ and $\Delta\varepsilon = 10^{-3}$.

n		$\varepsilon^* - \Delta\varepsilon$	ε^*	$\varepsilon^* + \Delta\varepsilon$
3		.69001	.68928	.68854
4		.73787	.73668	.73548
5		.71595	.71396	.71195
6		.73204	.72883	.72560
7		.72106	.71590	.71066
8		.73472	.72633	.71771
9		.73223	.71877	.70473
10	x_0^n	.74553	.72395	.70080
11		.75447	.72020	.68183
12		.77568	.72199	.65746
13		.80278	.72044	.60986
14		.84270	.72145	.52845
15		.88913	.72108	.38526
16		.93543	.72112	.18828
17		.96976	.72113	$.42202 \times 10^{-2}$
18		.98756	.72114	2.8981×10^{-3}
19		.99455	.72117	

Table VIII. Scaling factors β_n and α_n along and across the dominant symmetry line when $\varepsilon = \varepsilon^*$

n	β_n	α_n
4	−3.0326	−1.4232
5	−3.0725	−1.4082
6	−3.0469	−1.4191
7	−3.0735	−1.4128
8	−3.0630	−1.4174
9	−3.0658	−1.4138
10	−3.0660	−1.4164
11	−3.0651	−1.4146
12	−3.0669	−1.4157
13	−3.0673	−1.4147
14	−3.0680	−1.4150
15	−3.0661	−1.4148
16	−3.0670	−1.4149
17	−3.0670	−1.4148
18	−3.0669	−1.4148
19	−3.0668	−1.4149
20	−3.0668	−1.4149

case. When $\varepsilon > \varepsilon^*$, F_n approaches to some nonzero value. In this case, the invariant curve is broken into a cantorus. At $\varepsilon = \varepsilon^* + 10^{-3}$, the observed flux through the cantorus is 1.23×10^{-11}.

The θ-coordinate θ_n of the minimizing periodic point nearest to the dominant symmetry line approaches to zero when $\varepsilon \leq \varepsilon^*$. For the critical case, θ_n obeys a power law decay:

$$\theta_n \sim q_n^{-x_0}, \quad x_0 = 0.711$$

Therefore, the critical invariant curve is not differentiably but topologically conjugate to the uniform rotation. When $\varepsilon < \varepsilon^*$, the observed power x_0 seems to be nearly 1. For the supercritical case, θ_n approaches to some nonzero value. When $\varepsilon = \varepsilon^*$, the observed limiting value is 2.22×10^{-3}. Therefore, the invariant curve is broken into the cantrous with an infinity of gaps.

We now describe the scaling along and across the symmetry lines for the critical invariant curve. We use symmetry coordinates (X, Y). For S-symmetry, symmetry coordinates are: $X = \theta$ and $Y = I - \frac{\varepsilon}{2} \cdot F(\theta)$. For TS-symmetry, symmetry coordinates are : $X = \theta - 1/2$ and $Y = I$. In the symmetry coordinates, the symmetries are represented as $(X, Y) \rightarrow (X', Y') = (-X - n, Y)$; $n \in Z$.

Firstly, we describe the scaling behavior near the dominant half-line. We call the periodic point $(0, Y_n)$ on the dominant half-line the dominant point. We measured the position Y_n of the dominant point. $\{Y_n\}$- sequence converges geometrically to the invariant curve with a ratio β. The convergence ratio β is the limit value of $\{\beta_n\}$-sequence defined by $\beta_n = (Y_{n-1} - Y_n)/(Y_n - Y_{n-1})$. This sequence is shown in the TABLE VII. The observed scaling factor β along the dominant half-line is:

$$\beta = -3.068 .$$

Therefore, Y_n approaches to the invariant curve in a nonanalytic fashion:

$$\left| \frac{Y_{n-1} - Y_n}{Y_n - Y_{n-1}} \right| \sim \gamma^{-y_0} ; \quad y_0 = \log_\gamma |\beta| .$$

This is consistent with

$$| Y_n - Y^* | \sim q_n^{-y_0}$$

where Y^* is the limit value of $\{Y_n\}$-sequence. The observed value of Y^* is:

$$Y^* = .405611110478107.$$

The scaling behavior across the dominant half-line can be studied by measuring the positions (X_n, Y_n) of the nearest point of the periodic orbit to the dominant point. $\{X_n\}$ – sequence converges geometrically to the dominant half-line. The convergence ratio α is the limit value of $\{\alpha_n\}$-sequence defined by $\alpha_n = X_n/X_{n+1}$. This sequence is included in the TABLE VIII. The observed scaling factor α across the dominant half-line is:

$$\alpha = -1.4148.$$

This is consistent with

$$|X_n| \sim q_n^{-x_0} ; X_0 = \log_\gamma |\alpha| .$$

Secondly, we describe the scaling behaviors near the three subdominant half-lines. We call the periodic point on the subdominant half-line the subdominant point. In a similar way, by measuring the positions of the subdominant point and the nearest point to it, the scaling behaviors can be studied. But the scalings exhibit 'period-3' behaviors. These results are included in the tables (from TABLE IX to TABLE X). The 3-step scaling factors β_3 and α_3 along and across the dominant half-lines are:

Table IX. Scaling factors β_n along the three subdominant half lines when $\varepsilon = \varepsilon^*$

n	$\theta = 1/2$	$I = 2\theta$	$I = 2\theta - 1$
4	−2.0485	−3.3367	−2.4447
5	−3.3876	−2.4215	−2.0625
6	−2.4459	−2.0686	−3.3489
7	−2.0577	−3.3795	−2.4165
8	−3.3542	−2.4396	−2.0643
9	−2.4254	−2.0584	−3.3714
10 β_n	−2.0621	−3.3625	−2.04345
11	−3.3692	−2.4285	−2.0594
12	−2.4318	−2.0604	−3.3673
13	−2.0589	−3.3698	−2.4291
14	−3.3678	−2.4305	−2.0597
15	−2.4306	−2.596	−3.3678
16	−2.0596	−3.3684	−2.4302
17	−3.3682	−2.4303	−2.0596
18	−2.4303	−2.0597	−3.3681
19	−2.0597	−3.3679	−2.4304
20	−3.3679	−2.4303	−2.0597

Table X. Scaling factors α_n across the three subdominant half lines when $\varepsilon = \varepsilon^*$

n	$\theta = 1/2$	$I = 2\theta$	$I = 2\theta - 1$
4	−1.7835	−1.6283	−1.7055
5	−1.6101	−1.7087	−1.7706
6	−1.6966	−1.7734	−1.6075
7	−1.7729	−1.6005	−1.7067
8	−1.6022	−1.6980	−1.7748
9	−1.7065	−1.7722	−1.6038
10 α_n	−1.7747	−1.6029	−1.7013
11	−1.6039	−1.7042	−1.7735
12	−1.7025	−1.7745	−1.6030
13	−1.7743	−1.6033	−1.7037
14	−1.6032	−1.7030	−1.7746
15	−1.7034	−1.7741	−1.6037
16	−1.7743	−1.6034	−1.7033
17	−1.6034	−1.7032	−1.7744
18	−1.7033	−1.7743	−1.6035
19	−1.7743	−1.6034	−1.7033
20	−1.6034	−1.7033	−1.7743

$$\beta_3 = -16.859,$$

$$\alpha_3 = -4.8458.$$

Note that $\beta_3 \neq \beta^3$, $\alpha_3 \neq \alpha^3$ and $\alpha_3 \cdot \beta_3 = (\alpha \cdot \beta)^3$. Therefore, though the scalings along and across the symmetry half-lines exhibit different behaviors, the area-scalings exhibit the same behaviors. This is consistent with the fact that the ratio of the fluxes F_n/F_{n+1} approaches to some value $\xi : \xi = |\alpha\,\beta|$.

In this section, we showed numerically that for an area-preserving map of class-C^2, the invariant curve of winding number γ^{-1} persists below the critical parameter value, and the critical behaviors are the same as those in the standard map studied by Shenker and Kadanoff[12], and Mackay[11].

III. REVIVAL OF RATIONAL INVARIANT CURVES IN A PIECEWISE LINEAR MAP OF CLASS-C^0

We study a reversible, periodic, area-preserving twist map T of class$-C^0$:

$$T : \begin{cases} I' = I + \varepsilon F(\theta) \\ \theta' = \theta - I' \end{cases},$$

$$\text{where } F(\theta) = \begin{cases} -\theta, & 0 \leq \theta \leq 1/4 \\ \theta - 1/2, & 1/4 \leq \theta \leq 3/4 \\ 1 - \theta, & 3/4 \leq \theta \leq 1 \end{cases},$$

$$F(\theta) = F(\theta - 1) \text{ and } \int_0^1 F(\theta) \cdot d\theta = 0.$$

For the twist may T, the generating function $L(\theta, \theta')$ is:

$$L(\theta, \theta') = 1/2 \cdot (\theta - \theta')^2 - \varepsilon \cdot V(\theta),$$

$$\text{where } V(\theta) = \begin{cases} \theta^2/2 - 1/32, & 0 \leq \theta \leq 1/4 \\ -1/2 \cdot (\theta - 1/2)^2 + 1/32, & 1/4 \leq \theta \leq 3/4 \\ 1/2 \cdot (\theta - 1)^2 - 1/32, & 3/4 \leq \theta \leq 1 \end{cases}$$

Since the map T is a reversible map, T can be factored into the product $(TS)\cdot S$ of two involutions, where

$$S:\begin{cases}\theta' = -\theta + n\\ I' = I - \varepsilon\cdot F'(\theta)\end{cases};\ n\in z\ (\text{integer}).$$

The four symmetry lines formed from the invariant points under S and TS are $\theta = 0$, $\theta = 1/2$, $I = 2\theta$ and $I = 2\theta - 1$.

Since the smoothness of the map T is below the critical smoothness, all invariant curves are expected to be broken and extended chaos may occur for arbitrarily small ε. But Chirikov observed that at some parameter value global choas disappears and thus the chaotic component was confined in I.[1,3]

We show that this peculiar phenomenon can be explained by the fact that rational invariant curves revive at some ε. For this map T de la Llave has an example with a rational invariant curve of winding number 0 (referred to by Mackay)[11] when $\varepsilon = 4/3$. The rational invariant curve $C(0)$ of winding number 0 is:

$$C(0):\begin{cases} I = 2\theta/3 + 1/3, & 0\le\theta\le 1/4\\ I = -2\theta - 1, & 1/4\le\theta\le 1/2\\ I = 2\theta/3 - 1/3, & 1/2\le\theta\le 1\end{cases}$$

This saturates the bounds of Mather's result on non-existence of invariant curves and thus there are arbitrarily small perturbations with no invariant curves at all[11]. We also find rational invariant circles $C(1/3)$ and $C(1/4)$ of winding number 1/3 and 1/4. When $\varepsilon = 1$, the rational invariant curve $C(1/3)$ of winding number 1/3 revives, where

$$C(1/3):\begin{cases} I = \theta/2 + 3/8, & 0\le\theta\le 1/4\\ I = -\theta + 3/4, & 1/4\le\theta\le 1/2\\ I = 1/4, & 1/2\le\theta\le 3/4\\ I = \theta/2 - 1/8, & 3/4\le\theta\le 1,\end{cases}$$

and when $\varepsilon = \sqrt{5} - 1$

and the rational invariant curve $C(1/4)$ of winding 1/4 revives, where

$$C(1/4):\begin{cases} I = \gamma^{-1}(\theta-\frac{1}{4}) + \frac{1}{2}, & 0\le\theta\le 1/4\\ I = -\gamma(\theta-\frac{1}{4}) + \frac{1}{2}, & 1/4\le\theta\le(4-\sqrt{5})/4\\ I = -\gamma^{-1}(\theta-\frac{1}{2}) - \frac{\gamma^{-1}}{4}, & (4-\sqrt{5})/4\le\theta\le\sqrt{5}/4\\ I = \gamma^{-2}(\theta-\frac{3}{4}) - \gamma^{-2}/2, & \frac{\sqrt{5}}{4}\le\theta\le 3/4\\ I = \gamma^{-1}(\theta-\frac{3}{4})\gamma^{-2}/2, & 3/4\le\theta\le 1.\end{cases}$$

$\gamma = (1-\sqrt{5})/2.$

We find numerically rational invariant circles of winding number p/q $(q\ge 5)$. It is observed that when the parameter ε crosses some parameter value at which the θ-coordinate of a periodic point of one of the two minimizing and minimaximizing symmetric periodic orbits of winding number p/q crosses the $\theta = 1/4$ - line, an extremum transition in action occurs, such that the minimizing orbit turns to the minimaxizing one, and vice versa. When the extremum transition occurs, the action difference between the two symmetric periodic orbits becomes zero, and thus the rational invariant curve of winding number p/q revives. Extremum transitions occur $(2m-1)$ times when $q=4m$, $4m-1$ $(m=1, 2\ldots)$, and $2m$ times when $q=4m+1$, $4m+2$ $(m=0, 1\ldots)$. For example, for the two symmetric periodic orbits of winding number 1/5, the extremum transitions occur two times when $\varepsilon =$ .30277563, 1.30277564. It is also observed that

Table X. Final extremum transition values $\varepsilon^*(1/q)$ and the convergence ratio.

q	$\varepsilon^*(1/q)$	δ_q
3	1	3.42705098
4	$\sqrt{5}-1$	3.18300690
5	1.30277564	3.07760292
6	1.32340428	3.03214932
7	1.33005874	3.01297096
8	1.33224650	3.00510601
9	1.33297167	3.00196819
10	1.33321286	3.00074567
11	1.33329319	3.00027856
12	1.33331995	3.00010288
13	1.33332887	3.00003764
14	1.33333185	3.00001366
15	1.33333284	3.00000493
16	1.33333317	3.00000175
17	1.33333329	3.00000073
18	1.33333331	

the final extremum transition of the two symmetric periodic orbits of winding number p/q occurs at the same parameter value dependent only upon q irrespective of p. For example, the final extremum transitions of symmetric periodic orbits of winding number 1/7, 2/7 and 3/7 occur at the same time when $\varepsilon=1.33005874$. After the final extremum transition, all the periodic points of the two symmetric periodic orbits except one periodic point on the $\theta=0$– line lie between $\theta=1/4$ – line and $\theta=3/4$ – line.

We restrict our considerations of extremum transitions to the final transitions. The final extremum transition values $\varepsilon^*(1/q)$ of the two symmetric periodic orbits of winding number $1/q$ are shown in the TABLE XI. $\{\varepsilon^*(1/q)\}$ – sequence converges to some limit value with a geometric ratio δ. The limit value is 4/3, which is just the value at which the rational invariant curve of winding number 0 is found to exist by de la Llave. The ratio δ is the limit value of $\{\delta_p\}$ – sequence defined by $\delta_q=(4/3-\varepsilon^*(1/q))/(4/3-\varepsilon^*(1/(q-1)))$: $\delta=3$. $\{\delta_q\}$ – sequence is included in the TABLE XI. Therefore, when $\varepsilon>4/3$ no extremum transition occurs and extended chaos takes place. Even when ε is less than 4/3, global chaos can occur in the absence of a rational invariant curve.

IV. SUMMARY AND DISCUSSION

According to the pendulum approximation and Greene's residue criterion, the critical smoothness of perturbations is class-C^1 for the map. But these approximations are not very accurate. Thus, we study numerically whether or not a noble invariant circle persists under a perturbation of class-C^2. Following Greene's residue criterion, we show that the invariant curve of winding number γ^{-1} persists below the critical parameter value. Therefore, below the critical parameter value, the invariant curve plays the role of a complete barrier to the transport of stochastic orbits. It is also observed that the critical behaviors seem to be the same as those for analytic perturbations.

In a piecewise linear ma[illegible]'s observed that an extremum transition occurs between the two symmetric minimizing and minimaximizing periodic orbits. At the transition parameter value, the difference in actions is zero between the two periodic orbits and a rational invariant curve revives. In the piecewise linear map, the rational invariant curves play the role of barriers to transport. It is also observed that when $\varepsilon>4/3$, no extremum transition takes place. Therefore for any parameter value greater than 4/3, global chaos occurs. When $\varepsilon\leq 4/3$ both extended and confined chaos occur as the parameter varies, depending upon the existence of a rational invariant curve. It is still a mystery to us why an extremum transition takes place when the parameter crosses some value at which the θ-coordinate of one of the two periodic orbits crosses the $\theta = 1/4$ - line.

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면적이 보존되는 C^2 와 C^0 Class의 본뜨기에서의 수송 장벽

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C^0 와 C^2 Class의 면적이 보존되는 본뜨기를 수치적으로 연구하였다. C^2 Class의 경우 noble 불변곡선이 임계 섭동변수값 아래에서 존재하고 해석적 섭동의 경우와 그 임계동향이 서로 같음을 보였다. 그러나 C^0 Class의 본뜨기에서는 유리수적 불변곡선이 되살아 나서 멋대로 하는 운동을 한정시키는 실질적인 장벽 역할을 한다.