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M-FURCATIONS IN COUPLED MAPS

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We study the scaling behavior of M-furcation ($M = 2, 3, 4, \ldots$) sequences of M^n-periodic ($n = 1, 2, \ldots$) orbits in two coupled one-dimensional (1D) maps. Using a renormalization method, how the scaling behavior depends on M is particularly investigated in the zero-coupling case in which the two 1D maps become uncoupled. The zero-coupling fixed map of the M-furcation renormalization transformation is found to have three relevant eigenvalues δ, α, and M (δ and α are the parameter and orbital scaling factors of 1D maps, respectively). Here the second and third ones, α and M, called the "coupling eigenvalues", govern the scaling behavior of the coupling parameter, while the first one δ governs the scaling behavior of the nonlinearity parameter like in the case of 1D maps. The renormalization results are also confirmed by a direct numerical method.

1. Introduction

Universal scaling behaviors of M-furcation ($M = 2, 3, 4, \ldots$) sequences of M^n-cycles ($n = 1, 2, \ldots$) (i.e. M^n-periodic orbits) have been found in a one-parameter family of one-dimensional (1D) unimodal maps with a quadratic maximum. As an example, consider the logistic map

$$x_{t+1} = f(x_t) = 1 - A\,x_t^2, \tag{1}$$

where t denotes the discrete time. As the nonlinearity parameter A is increased from 0, a stable fixed point undergoes the cascade of period-doubling bifurcations accumulating at a finite parameter value $A_\infty (= 1.401\,155\ldots)$. The bifurcation sequence corresponding to the MSS (Metropolis, Stein, and Stein[1]) sequence R^{*n} (for details of the MSS sequences and the $(*)$-composition rule, see Refs. 1 and 2) exhibits an asymptotic scaling behavior.[3]

What happens beyond the bifurcation accumulation point A_∞ is interesting from the viewpoint of chaos. The parameter interval between A_∞ and the final boundary-crisis point $A_c (= 2)$ beyond which no periodic or chaotic attractors can be found within the unimodality interval is called the "chaotic" regime. Within this region, besides the bifurcation sequence, there are many other sequences of periodic orbits exhibiting their own scaling behaviors. In particular, every primary pattern P (that cannot be decomposed using the $(*)$-operation) leads to an MSS sequence

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P^{*n}. For example, $P = RL$ leads to a trifurcation sequence of 3^n-cycles, $P = RL^2$ to a tetrafurcation sequence of 4^n-cycles, and the three different $P = RLR^2$, RL^2R, and RL^3 to three different period-5^n sequences. Thus there exist infinitely many higher M-furcation ($M = 3, 4, \dots$) sequences inside the chaotic regime. Unlike the bifurcation sequence, stability regions of periodic orbits in the higher M-furcation sequences are not adjacent on the parameter axis, because they are born by their own tangent bifurcations. The asymptotic scaling behaviors of these (disconnected) higher M-furcation sequences characterized by the parameter and orbital scaling factors, δ and α, vary depending on the primary pattern P.[2,4–11]

In this paper we consider two symmetrically coupled 1D maps. This coupled map may help us understand how coupled oscillators, such as Josephson-junction arrays or chemically reacting cells, exhibit various dynamical behaviors.[12–14] We are interested in the scaling behavior of M-furcations ($M = 2, 3, \dots$) in the two coupled 1D maps. The bifurcation case ($M = 2$) was previously studied in Refs. 15–20. Here we extend the result for the bifurcation case to all the other higher multifurcation cases with $M = 3, 4, \dots$ in the zero-coupling case where the two 1D maps become uncoupled. In Sec. 2 we investigate the dependence of the scaling behavior on M using a renormalization method. It is found that the zero-coupling fixed point of the M-furcation renormalization transformation has three relevant eigenvalues δ, α and M. The scaling behavior of the coupling parameter is governed by two coupling eigenvalues (CE's) α and M, while the scaling behavior of the nonlinearity parameter is governed by the eigenvalue δ like in the case of 1D maps. As an example, we numerically study the scaling behavior of the coupling parameter in the trifurcation sequence in Sec. 3 and confirm the renormalization results. Finally, a summary is given in Sec. 4.

2. Renormalization Analysis

In this section we first introduce two coupled 1D maps and discuss stability of orbits, and then study the scaling behavior of M-furcations ($M = 2, 3, \dots$) in the zero-coupling case using the renormalization method developed in Refs. 15 and 19. It is found that there exist three relevant eigenvalues δ, α and M. As in the case of 1D maps, the scaling behavior of the nonlinearity parameter is governed by the eigenvalue δ, irrepectively of coupling. However, the scaling behavior of the coupling parameter depends on the nature of coupling. In a linear-coupling case, in which the coupling function has a leading linear term, it is governed by two CE's, α and M, whereas it is governed by only one CE, M, in the other cases of nonlinear coupling with leading nonlinear terms.

Consider a map T consisting of two symmetrically coupled 1D maps,

$$T : \begin{cases} x_{t+1} = F(x_t, y_t) = f(x_t) + g(x_t, y_t), \\ y_{t+1} = F(y_t, x_t) = f(y_t) + g(y_t, x_t), \end{cases} \tag{2}$$

where $f(x)$ is a 1D unimodal map with a quadratic maximum at $x = 0$, and $g(x, y)$

is a coupling function. The uncoupled 1D map f satisfies a normalization condition

$$f(0) = 1, \tag{3}$$

and the coupling function g obeys a condition

$$g(x, x) = 0 \quad \text{for any } x. \tag{4}$$

The two-coupled map (2) is invariant under the exchange of coordinates such that $x \leftrightarrow y$. The set of all points which are invariant under the exchange of coordinates forms a symmetry line $y = x$. An orbit is called an "in-phase" orbit if it lies on the symmetry line, i.e. it satisfies

$$x_t = y_t \quad \text{for all } t. \tag{5}$$

Otherwise it is called an "out-of-phase" orbit. Here we study only in-phase orbits, which can be easily found from the uncoupled 1D map, $x_{t+1} = f(x_t)$, because of the condition (4).

Stability of an in-phase orbit with period p is determined from the Jacobian matrix J of T^p, which is the p-product of the Jacobian matrix DT of T along the orbit:

$$J = \prod_{t=1}^{p} DT(x_t, x_t) = \prod_{t=1}^{p} \begin{pmatrix} f'(x_t) - G(x_t) & G(x_t) \\ G(x_t) & f'(x_t) - G(x_t) \end{pmatrix}, \tag{6}$$

where the prime denotes a derivative, and $G(x) = \partial g(x, y)/\partial y \mid_{y=x}$; hereafter, $G(x)$ will be referred to as the "reduced coupling function" of $g(x, y)$. The eigenvalues of J, called the stability multipliers of the orbit, are

$$\lambda_1 = \prod_{t=1}^{p} f'(x_t), \quad \lambda_2 = \prod_{t=1}^{p} [f'(x_t) - 2G(x_t)]. \tag{7}$$

Note that the first stability multiplier λ_1 is just that of the uncoupled 1D map and the coupling affects only the second stability multiplier λ_2.

An in-phase orbit is stable only when the moduli of both multipliers are less than or equal to unity, i.e. $-1 \le \lambda_i \le 1$ for $i = 1, 2$. A tangent (period-doubling) bifurcation occurs when each stability multiplier λ_i increases (decreases) through 1 (-1). Hence the stable region of the in-phase orbit in the parameter plane is bounded by four bifurcation lines associated with tangent and period-doubling bifurcations (i.e. those curves determined by the equations $\lambda_i = \pm 1$ for $i = 1, 2$).

We now consider the M-furcation renormalization transformation $\mathcal{N}$, which is composed of the M-times iterating ($T^{(M)}$) and rescaling (B) operators

$$\mathcal{N}(T) \equiv BT^{(M)}B^{-1}. \tag{8}$$

Here the rescaling operator B is

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \tag{9}$$

because we consider only in-phase orbits.

Applying the renormalization operator $\mathcal{N}$ to the coupled map (2) n times, we obtain the n-times renormalized map T_n of the form

$$T_n : \begin{cases} x_{t+1} = F_n(x_t, y_t) = f_n(x_t) + g_n(x_t, y_t), \\ y_{t+1} = F_n(y_t, x_t) = f_n(y_t) + g_n(y_t, x_t). \end{cases} \tag{10}$$

Here f_n and g_n are the uncoupled and coupling parts of the n-times renormalized function F_n, respectively. They satisfy the following recurrence equations

$$f_{n+1}(x) = \alpha f_n^{(M)}\left(\frac{x}{\alpha}\right), \tag{11}$$

$$g_{n+1}(x, y) = \alpha F_n^{(M)}\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) - \alpha f_n^{(M)}\left(\frac{x}{\alpha}\right), \tag{12}$$

where $f_n^{(M)}(x) = f_n(f_n^{(M-1)}(x))$ and $F_n^{(M)}(x, y) = F_n(F_n^{(M-1)}(x, y), F_n^{(M-1)}(y, x))$.

The recurrence relations (11) and (12) define a renormalization operator $\mathcal{R}$ of transforming a pair of functions (f, g)

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = \mathcal{R} \begin{pmatrix} f_n \\ g_n \end{pmatrix}. \tag{13}$$

The renormalization transformation $\mathcal{R}$ obviously has a fixed point (f^*, g^*) with $g^*(x, y) = 0$, which satisfies $\mathcal{R}(f^*, 0) = (f^*, 0)$. Here f^* is just the 1D fixed function satisfying

$$f^*(x) = \alpha f^{*(M)}\left(\frac{x}{\alpha}\right), \tag{14}$$

where $\alpha = 1/f^{*(M-1)}(1)$, due to the normalization condition (3), and it has the form

$$f^*(x) = 1 + c_1^* x^2 + c_2^* x^4 + \cdots, \tag{15}$$

where c_i^*'s $(i = 1, 2, \dots)$ are some constants. The fixed point $(f^*, 0)$ governs the critical behavior near the zero-coupling critical point because the coupling fixed function is identically zero, i.e. $g^*(x, y) = 0$. Here we restrict our attention to this zero-coupling case.

Consider an infinitesimal perturbation (h, φ) to the zero-coupling fixed point $(f^*, 0)$. We then examine the evolution of a pair of functions $(f^* + h, \varphi)$ under $\mathcal{R}$. Linearizing $\mathcal{R}$ at the zero-coupling fixed point, we obtain a linearized operator $\mathcal{L}$ of transforming a pair of perturbations (h, ϕ)

$$\begin{pmatrix} h_{n+1} \\ \varphi_{n+1} \end{pmatrix} = \mathcal{L} \begin{pmatrix} h_n \\ \varphi_n \end{pmatrix} = \begin{pmatrix} \mathcal{L}_u & 0 \\ 0 & \mathcal{L}_c \end{pmatrix} \begin{pmatrix} h_n \\ \varphi_n \end{pmatrix}, \tag{16}$$

where

$$h_{n+1}(x) = [\mathcal{L}_u h_n](x) \tag{17}$$

$$= \alpha\,\delta f_n^{(M)}\left(\frac{x}{\alpha}\right) \equiv \alpha\left[f_n^{(M)}\left(\frac{x}{\alpha}\right) - f_n^{*(M)}\left(\frac{x}{\alpha}\right)\right]_{\text{linear}} \tag{18}$$

$$= \alpha f^{*\prime}\left(f^{*(M-1)}\left(\frac{x}{\alpha}\right)\right)\delta f_n^{(M-1)}\left(\frac{x}{\alpha}\right) + \alpha h_n\left(f^{*(M-1)}\left(\frac{x}{\alpha}\right)\right), \tag{19}$$

$$\varphi_{n+1}(x,y) = [\mathcal{L}_c\varphi_n](x,y) \tag{20}$$

$$= \alpha\,\delta\left[F_n^{(M)}\left(\frac{x}{\alpha},\frac{y}{\alpha}\right) - f_n^{(M)}\left(\frac{x}{\alpha}\right)\right] \equiv \alpha\left[F_n^{(M)}\left(\frac{x}{\alpha},\frac{y}{\alpha}\right)\right.$$

$$\left. - f_n^{(M)}\left(\frac{x}{\alpha}\right)\right]_{\text{linear}} \tag{21}$$

$$= \alpha f^{*\prime}\left(f^{*(M-1)}\left(\frac{x}{\alpha}\right)\right)\delta\left[F_n^{(M-1)}\left(\frac{x}{\alpha},\frac{y}{\alpha}\right) - f_n^{(M-1)}\left(\frac{x}{\alpha}\right)\right]$$

$$+\alpha\varphi_n\left(f^{*(M-1)}\left(\frac{x}{\alpha}\right), f^{*(M-1)}\left(\frac{y}{\alpha}\right)\right). \tag{22}$$

Here the variations $\delta f_n^{(M)}(\frac{x}{\alpha})$ and $\delta[F_n^{(M)}(\frac{x}{\alpha},\frac{y}{\alpha}) - f_n^{(M)}(\frac{x}{\alpha})]$ are introduced as the linear terms (denoted by $[f_n^{(M)}(\frac{x}{\alpha}) - f_n^{*(M)}(\frac{x}{\alpha})]_{\text{linear}}$ and $[F_n^{(M)}(\frac{x}{\alpha},\frac{y}{\alpha}) - f_n^{(M)}(\frac{x}{\alpha})]_{\text{linear}}$) in h and φ of the deviations of $f_n^{(M)}(\frac{x}{\alpha})$ and $F_n^{(M)}(\frac{x}{\alpha},\frac{y}{\alpha}) - f_n^{(M)}(\frac{x}{\alpha})$ from $f^{*(M)}(\frac{x}{\alpha})$ and 0, respectively. A pair of perturbations (h^*, φ^*) is then called an eigenperturbation with eigenvalue ν, if it satisfies

$$\nu\begin{pmatrix} h^* \\ \varphi^* \end{pmatrix} = \mathcal{L}\begin{pmatrix} h^* \\ \varphi^* \end{pmatrix}, \tag{23}$$

i.e.

$$h^*(x) = [\mathcal{L}_u h^*](x), \tag{24}$$

$$\varphi^*(x,y) = [\mathcal{L}_c\varphi^*](x,y). \tag{25}$$

The eigenperturbations of the linear operator $\mathcal{L}$ can be divided into two classes. The first class of eigenperturbations are of the form $(h^*, 0)$. Here $h^*(x)$ is an eigenfunction of the linear "uncoupled operator" $\mathcal{L}_u$ satisfying Eq. (24), which is just the eigenvalue equation in the uncoupled 1D case. It has been found in Refs. 5, 6 and 8 that there exists a unique eigenfunction $h^*(x)$ with (noncoordinate change) relevant eigenvalue δ, associated with scaling of the nonlinearity parameter.

The second class of eigenperturbations have the form $(0, \varphi^*)$, where $\varphi^*(x)$ is an eigenfunction of the linear "coupling operator" $\mathcal{L}_c$ satisfying Eq. (25). However, it is not easy to directly solve the coupling eigenvalue equation (25). We therefore introduce a tractable recurrence equation for a "reduced coupling eigenfunction " of $\varphi^*(x,y)$, defined by

$$\Phi^*(x) \equiv \left.\frac{\partial\varphi^*(x,y)}{\partial y}\right|_{y=x}. \tag{26}$$

Differentiating Eq. (25) with respect to y and setting $y = x$, we obtain an eigenvalue equation for a reduced linear coupling operator $\tilde{\mathcal{L}}_c$

$$\nu\, \Phi^*(x) = [\tilde{\mathcal{L}}_c \Phi^*](x) \tag{27}$$
$$= \delta F_2^{(M)}\left(\frac{x}{\alpha}\right) = \left[F_2^{(M)}\left(\frac{x}{\alpha}\right)\right]_{\rm linear} \tag{28}$$
$$= f^{*\prime}\left(f^{*(M-1)}\left(\frac{x}{\alpha}\right)\right) \delta F_2^{(M-1)}\left(\frac{x}{\alpha}\right) + f^{*(M-1)\prime}\left(\frac{x}{\alpha}\right) \Phi^*\left(f^{*(M-1)}\left(\frac{x}{\alpha}\right)\right). \tag{29}$$

Here $F(x,y) = f^*(x) + \varphi^*(x,y)$, $F_2^{(M)}(x)$ is a "reduced function" of $F^{(M)}(x,y)$ defined by $F_2^{(M)}(x) \equiv \partial F^{(M)}(x,y)/\partial y|_{y=x}$, and the variation $\delta F_2^{(M)}(\frac{x}{\alpha})$ is also introduced as the linear term (denoted by $[F_2^{(M)}(\frac{x}{\alpha})]_{\rm linear}$) in Φ^* of the deviation of $F_2^{(M)}(\frac{x}{\alpha})$ from 0.

In the case $M = 2$, the variation $\delta F_2^{(2)}(\frac{x}{\alpha})$ of Eq. (28) becomes

$$\delta F_2^{(2)}\left(\frac{x}{\alpha}\right) = \Phi^*\left(\frac{x}{\alpha}\right) f^{*\prime}\left(f^*\left(\frac{x}{\alpha}\right)\right) + f^{*\prime}\left(\frac{x}{\alpha}\right) \Phi^*\left(f^*\left(\frac{x}{\alpha}\right)\right). \tag{30}$$

Substituting $\delta F_2^{(2)}(\frac{x}{\alpha})$ into Eq. (29), we have $\delta F_2^{(3)}(\frac{x}{\alpha})$ for $M = 3$, which consists of three terms,

$$\begin{aligned}\delta F_2^{(3)}\left(\frac{x}{\alpha}\right) = {} & \Phi^*\left(\frac{x}{\alpha}\right) f^{*\prime}\left(f^*\left(\frac{x}{\alpha}\right)\right) f^{*\prime}\left(f^{*(2)}\left(\frac{x}{\alpha}\right)\right) \\ & + f^{*\prime}\left(\frac{x}{\alpha}\right) \Phi^*\left(f^*\left(\frac{x}{\alpha}\right)\right) f^{*\prime}\left(f^{*(2)}\left(\frac{x}{\alpha}\right)\right) \\ & + f^{*\prime}\left(\frac{x}{\alpha}\right) f^{*\prime}\left(f^*\left(\frac{x}{\alpha}\right)\right) \Phi^*\left(f^{*(2)}\left(\frac{x}{\alpha}\right)\right).\end{aligned} \tag{31}$$

Repeating this procedure sucessively, we obtain $\delta F_2^{(M)}\left(\frac{x}{\alpha}\right)$ for a general M, composed of M terms,

$$\begin{aligned}\delta F_2^{(M)}\left(\frac{x}{\alpha}\right) &= \sum_{i=0}^{M-1} f^{*(i)\prime}\left(\frac{x}{\alpha}\right) \Phi^*\left(f^{*(i)}\left(\frac{x}{\alpha}\right)\right) f^{*(M-i-1)\prime}\left(f^{*(i+1)}\left(\frac{x}{\alpha}\right)\right) \\ &= \Phi^*\left(\frac{x}{\alpha}\right) f^{*(M-1)\prime}\left(f^*\left(\frac{x}{\alpha}\right)\right) + \cdots \\ &\quad + f^{*(i)\prime}\left(\frac{x}{\alpha}\right) \Phi^*\left(f^{*(i)}\left(\frac{x}{\alpha}\right)\right) f^{*(M-i-1)\prime}\left(f^{*(i+1)}\left(\frac{x}{\alpha}\right)\right) + \cdots \\ &\quad + f^{*(M-1)\prime}\left(\frac{x}{\alpha}\right) \Phi^*\left(f^{*(M-1)}\left(\frac{x}{\alpha}\right)\right),\end{aligned} \tag{32}$$

where $f^{(0)}(x) = x$.

Using the fact that $f^{*\prime}(0) = 0$, it can be easily shown that when $x = 0$, the reduced coupling eigenvalue equation (29) becomes

$$\nu\, \Phi^*(0) = \left[\prod_{i=1}^{M-1} f^{*\prime}(f^{*(i)}(0))\right] \Phi^*(0). \tag{33}$$

Differentiating the 1D fixed-point equation (14) with respect to x and then letting $x \to 0$, we also have

$$\prod_{i=1}^{M-1} f^{*\prime}(f^{*(i)}(0)) = \lim_{x\to 0} \frac{f^{*\prime}(x)}{f^{*\prime}(\frac{x}{\alpha})} = \alpha. \tag{34}$$

Then Eq. (33) reduces to

$$\nu\,\Phi^*(0) = \alpha\Phi^*(0). \tag{35}$$

There are two cases. If the coupling eigenfunction $\varphi^*(x,y)$ has a leading linear term, its reduced coupling eigenfunction $\Phi^*(x)$ becomes nonzero at $x = 0$. For this case $\Phi^*(0) \neq 0$, we have the first CE

$$\nu_1 = \alpha. \tag{36}$$

The eigenfunction $\Phi_1^*(x)$ with CE ν_1 is of the form

$$\Phi_1^*(x) = 1 + a_1^* x + a_2^* x^2 + \cdots, \tag{37}$$

where a_i^*'s $(i = 1, 2, \ldots)$ are some constants. For the other case $\Phi^*(0) = 0$, it is found that $f^{*\prime}(x)$ is an eigenfunction for the reduced coupling eigenvalue equation (29). Since Eq. (32) for the case $\Phi^*(x) = f^{*\prime}(x)$ becomes

$$\delta F_2^{(M)}\left(\frac{x}{\alpha}\right) = M f^{*(M)\prime}\left(\frac{x}{\alpha}\right), \tag{38}$$

the reduced coupling eigenvalue equation (29) reduces to

$$\nu f^{*\prime}(x) = M f^{*\prime}(x). \tag{39}$$

We therefore have the second relevant CE

$$\nu_2 = M, \tag{40}$$

with reduced coupling eigenfunction

$$\Phi_2^*(x) = f^{*\prime}(x). \tag{41}$$

Note that $\Phi_2^*(x)$ has no constant term, while $\Phi_1^*(x)$ has a constant term.

It is also found that there exists an infinite number of additional (coordinate change) reduced eigenfunctions $f^{*\prime}(x)\,[f^{*l}(x) - x^l]$ with irrelevant CE's α^{-l} $(l = 1, 2, \ldots)$, which are associated with coordinate changes.[16,19] We conjecture that together with the two (noncoordinate change) relevant CE's ($\nu_1 = \alpha$, $\nu_2 = M$), they give the whole spectrum of the reduced linear coupling operator $\tilde{\mathcal{L}}_c$ of Eq. (27) and the spectrum is complete.

In order to examine the effect of CE's on the M-furcation sequences, we consider an infinitesimal coupling perturbation

$$g(x,y) = c\,\varphi(x,y) \tag{42}$$

to a critical map at the zero-coupling critical point, in which case the two-coupled map has the form

$$T : \begin{cases} x_{t+1} = F(x_t, y_t) = f_{A_\infty^{(M)}}(x_t) + g(x_t, y_t), \\ y_{t+1} = F(y_t, x_t) = f_{A_\infty^{(M)}}(y_t) + g(y_t, x_t), \end{cases} \tag{43}$$

where $A_\infty^{(M)}$ denotes the accumulation value of the parameter A for the M-furcation case, and c is an infinitesimal coupling parameter. The map T at $c = 0$ is just the zero-coupling critical map consisting of two uncoupled 1D critical maps. It is attracted to the zero-coupling fixed map T_0^* consisting of two uncoupled 1D fixed maps, i.e.

$$T_0^* : x_{t+1} = f^*(x_t), \;\; y_{t+1} = f^*(y_t), \tag{44}$$

under iterations of the M-furcation renormalization transformation $\mathcal{N}$ of Eq. (8).

The reduced coupling function $G(x)$ of $g(x, y)$ is given by (see Eq. (26))

$$G(x) = c\,\Phi(x) \equiv c \left. \frac{\partial \varphi(x, y)}{\partial y} \right|_{y=x}. \tag{45}$$

We choose monomials x^l ($l = 0, 1, 2, \dots$) as initial reduced functions $\Phi(x)$, because any smooth function $\Phi(x)$ can be represented as a linear combination of monomials by a Taylor series. Expressing $\Phi(x) = x^l$ as a linear combination of eigenfunctions of $\tilde{\mathcal{L}}_c$, we have

$$\Phi(x) = x^l = \alpha_1 \Phi_1^*(x) + \alpha_2 f^{*\prime}(x) + \sum_{l=1}^{\infty} \beta_l f^{*\prime}(x)[f^{*l}(x) - x^l], \tag{46}$$

where α_1 and α_2 are relevant components, and all β_l's are irrelevant ones. The nth image Φ_n of Φ under the reduced linear coupling operator $\tilde{\mathcal{L}}_c$ of Eq. (27) has the form

$$\begin{aligned} \Phi_n(x) &= [\tilde{\mathcal{L}}_c^n \Phi](x) \\ &\simeq \alpha_1 \nu_1^n \Phi_1^*(x) + \alpha_2 \nu_2^n f^{*\prime}(x) \;\; \text{for large } n, \end{aligned} \tag{47}$$

because the irrelevant part of Φ_n becomes negligibly small for large n.

A coupling is called linear or nonlinear according to its leading term. In the case of a linear coupling, in which the coupling function $\varphi(x, y)$ has a leading linear term, its reduced coupling function $\Phi(x)$ has a leading constant term (e.g. $\Phi(x) = 1$ corresponds to a linear coupling case). However, for any other case of nonlinear coupling with a leading nonlinear term, its reduced coupling function contains no constant term (e.g. $\Phi(x) = x^l$ ($l \geq 1$) corresponds to a nonlinear coupling case). Hence the first relevant component α_1 becomes zero for $l \geq 1$, although it is nonzero for $l = 0$. It is therefore expected that the scalig behavior of the coupling parameter may be governed by the two relevant CEs $\nu_1 = \alpha$ and $\nu_2 = M$ for the linear-coupling case ($l = 0$), but by only the second relevant CE $\nu_2 = M$ for the other nonlinear-coupling cases ($l \geq 1$).

3. Numerical Analysis

Taking the trifurcation case with $M = 3$ as an example, we numerically study the scaling behavior of the coupling parameter in the two coupled 1D maps (43) with $f(x) = 1 - Ax^2$ and $g(x,y)$ $[= c\ \varphi(x)] = \frac{c}{m}(y^m - x^m)$ $(m = 1, 2, \dots)$. In this trifurcation case, we consider two kinds of coupling, linear and nonlinear coupling cases, and confirm the renormalization results.

As an example of the linear-coupling case, we consider linearly coupled maps, i.e. $g(x,y) = c\ (y - x)$. Figure 1 shows the stability regions of in-phase orbits with period $p = 3^n$ $(n = 1, 2, 3)$ near the $c = 0$ line. Note that Figs. 1(a), 1(b) and 1(c) nearly coincide near the zero-coupling point except for small numerical differences. It is therefore expected that the height and width h_n and w_n of the stability region of level n may geometrically contract in the limit of large n,

$$h_n \sim \delta^{-n}, \quad w_n \sim \alpha^{-n} \quad \text{for large } n, \tag{48}$$

where $\delta = 55.247\,026\dots$ and $\alpha = -9.277\,341\dots$. Hereafter the sequence of these stability regions will be referred to as a "trifurcation route".

We also follow the in-phase orbits of period $p = 3^n$ up to level $n = 10$ in the trifurcation route, and obtain a self-similar sequence of parameters (A_n, c_n), at which the orbit of level n has some given stability multipliers (λ_1, λ_2) (e.g. $\lambda_1 = -1$ and $\lambda_2 = 1$). Then the sequence $\{(A_n, c_n)\}$ converges geometrically to the zero-coupling critical point $(A^*, 0)$ $(A^* \equiv A_\infty^{(3)} = 1.786\,440\,255\,563\,639\,354\,534\,447\dots)$. In order to see the convergence of each of the two scalar sequences $\{A_n\}$ and $\{c_n\}$, define $\delta_n \equiv \frac{\Delta A_{n+1}}{\Delta A_n}$ and $\mu_n \equiv \frac{\Delta c_{n+1}}{\Delta c_n}$, where $\Delta A_n = A_n - A_{n-1}$ and $\Delta c_n = c_n - c_{n-1}$. Then they converge to their limit values δ and μ respectively, as shown in Table 1. Hence the two sequences $\{A_n\}$ and $\{c_n\}$ obey one-term scaling laws asymptotically

$$\Delta A_n \sim \delta^{-n}, \quad \Delta c_n \sim \mu^{-n} \quad \text{for large } n, \tag{49}$$

where $\delta = 55.247\dots$ and $\mu = -9.277\dots$. Note that the value of the coupling-parameter scaling factor μ is close to that of the first CE ν_1 $(= \alpha)$.

In order to take into account the effect of the second relevant CE ν_2 $(= 3)$ on the scaling of the sequence $\{\Delta c_n\}$, we extend the simple one-term scaling law (49) to a two-term scaling law [21]

$$\Delta c_n \sim C_1 \mu_1^{-n} + C_2 \mu_2^{-n} \quad \text{for large } n, \tag{50}$$

where $|\mu_2| > |\mu_1|$, and C_1 and C_2 are some constants. This is a kind of multiple scaling law.[22] Eq. (50) gives

$$\Delta c_n = t_1 \Delta c_{n+1} - t_2 \Delta c_{n+2}, \tag{51}$$

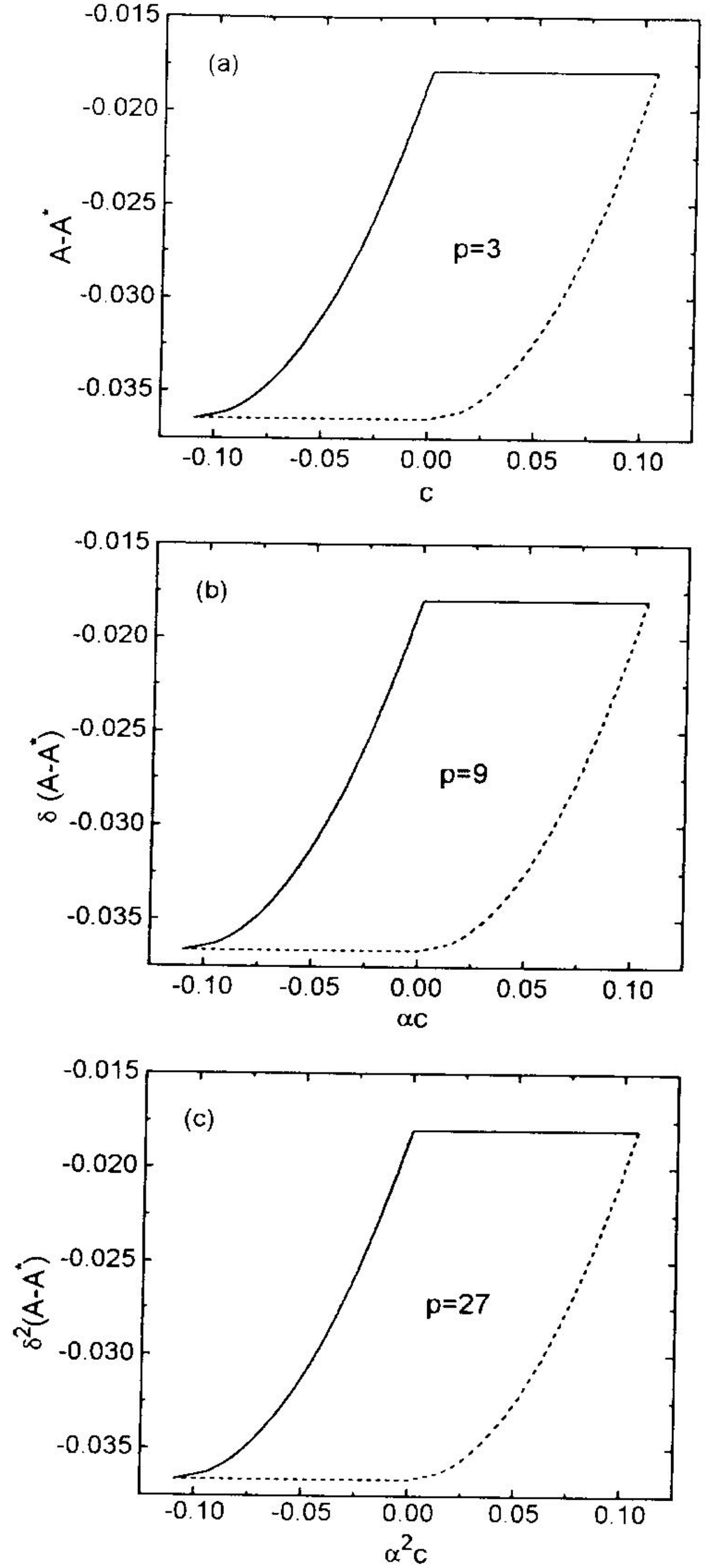

Fig. 1. Stable regions of the in-phase orbits with period $p = 3^n$ $(n = 1, 2, 3)$ in two linearly coupled 1D maps. The cases $n = 1, 2$, and 3 are shown in (a), (b) and (c), respectively. Each stable region is bounded by solid period-doubling bifurcation curves and dashed tangent bifurcation curves. The scaling factors used in (b) and (c) are $\delta = 55.247\,026$ and $\alpha = -9.277\,341$.

Table 1. In a linear-coupling case, we followed a sequence of parameters (A_n, c_n) at which the pair of stability multipliers $(\lambda_{1,n}, \lambda_{2,n})$ of the orbit of level n (period 3^n) is $(-1, 1)$. This sequence converges to the zero-coupling critical point $(A^*, 0)$ with the scaling factors shown in the second and third columns.

n	δ_n	μ_n
3	55.264 789 71	−9.272 61
4	55.245 771 51	−9.279 20
5	55.247 110 93	−9.276 71
6	55.247 020 84	−9.277 55
7	55.247 026 98	−9.277 27
8	55.247 026 56	−9.277 36
9	55.247 026 59	−9.277 33

where $t_1 = \mu_1 + \mu_2$ and $t_2 = \mu_1\mu_2$. Then μ_1 and μ_2 are solutions of the following quadratic equation

$$\mu^2 - t_1\mu + t_2 = 0. \tag{52}$$

To evaluate μ_1 and μ_2, we first obtain t_1 and t_2 from Δc_n's using Eq. (51)

$$t_1 = \frac{\Delta c_n \Delta c_{n+1} - \Delta c_{n-1}\Delta c_{n+2}}{\Delta c_{n+1}^2 - \Delta c_n \Delta c_{n+2}}, \tag{53a}$$

$$t_2 = \frac{\Delta c_n^2 - \Delta c_{n+1}\Delta c_{n-1}}{\Delta c_{n+1}^2 - \Delta c_n \Delta c_{n+2}}. \tag{53b}$$

Note that Eqs. (50)–(53) hold only for large n. In fact the values of t_i's and μ_i's $(i = 1, 2)$ depend on the level n. Therefore we explicitly denote t_i's and μ_i's by $t_{i,n}$'s and $\mu_{i,n}$'s respectively. Then each of them converges to a constant as $n \to \infty$:

$$\lim_{n\to\infty} t_{i,n} = t_i, \quad \lim_{n\to\infty} \mu_{i,n} = \mu_i, \quad i = 1, 2. \tag{54}$$

Three sequences $\{\mu_{1,n}\}$, $\{\mu_{2,n}\}$, and $\{\mu_{1,n}^2/\mu_{2,n}\}$ are shown in Table 2. The second column shows rapid convergence of $\mu_{1,n}$ to its limit values μ_1 $(= -9.277\,341\ldots)$, which is close to the renormalization result of the first relevant CE ν_1 $(= \alpha)$. (Its convergence to α is faster than that for the case of the above one-term scaling law.) From the third and fourth columns, we also find that the second scaling factor μ_2 is given by a product of two relevant CE's ν_1 and ν_2

$$\mu_2 = \frac{\nu_1^2}{\nu_2}, \tag{55}$$

where $\nu_1 = \mu_1$ and $\nu_2 = 3$. It has been known that every scaling factor in the multiple-scaling expansion of a parameter is expressed by a product of the eigenvalues of a linearized renormalization operator.[22]

Table 2. Scaling factors $\mu_{1,n}$ and $\mu_{2,n}$ in the two-term scaling for the coupling parameter are shown in the second and third columns, respectively. A product of them, $\frac{\mu_{1,n}^2}{\mu_{2,n}}$, is shown in the fourth column.

n	$\mu_{1,n}$	$\mu_{2,n}$	$\frac{\mu_{1,n}^2}{\mu_{2,n}}$
4	−9.277 396 06	24.578 89	3.501 79
5	−9.277 337 31	27.768 52	3.099 52
6	−9.277 341 36	28.502 38	3.019 72
7	−9.277 341 10	28.650 77	3.004 08
8	−9.277 341 12	28.681 53	3.000 85

The nonlinear-coupling case is studied with an example of quadratically coupled maps, i.e. $g(x,y) = \frac{c}{2}(y^2 - x^2)$. From Eq. (7), in this quadratic-coupling case the stability multipliers of in-phase orbits with period $p = 3^n$ become

$$\lambda_{1,n}(A) = \prod_{t=1}^{p} f'(x_t), \quad \lambda_{2,n}(A,c) = \left(1 + \frac{c}{A}\right)^p \lambda_{1,n}. \tag{56}$$

Let the pair of stability multipliers at a point (A_n, c_n) be $(\lambda_{1,n}, \lambda_{2,n})$. Then there exists a "conjugate point" $(A_n, -c_n - 2A_n)$, at which the pair of stability multipliers becomes $(\lambda_{1,n}, -\lambda_{2,n})$. For $c_n = -A_n$, $\lambda_{2,n} = 0$ and the two conjugate points become degenerate.

Figure 2 shows the stability diagram for the quadratic-coupling case. Like the linearly-coupled case, we also follow a self-similar sequence of parameters (A_n, c_n), at which the orbit of level n has some given stability multipliers (λ_1, λ_2). Without loss of generality we choose $\lambda_1 = -1$. Then one can find a pair of mutually conjugate sequences, depending on the value of λ_2. One sequence $\{(A_n, c_n)\}$ can be obtained by fixing $-1 < \lambda_2 < 0$, which converges to the zero-coupling critical point $(A^*, 0)$. Its "conjugate sequence" can also be obtained by following period-3^n orbits with $\lambda_{2,n} = -\lambda_2$. This kind of conjugate sequence $\{(A_n, -c_n - 2A_n)\}$ converges to the other critical point $(A^*, -2\,A^*)$ (i.e. the conjugate point of the zero-coupling critical point).

As an example, consider the case $\lambda_2 = -0.5$. The two conjugate sequences are denoted by the open up- and down-triangles in Fig. 2. We first consider the convergence of the sequence denoted by the uptriangles converging to the zero-coupling critical point. As shown in Table 3, the two sequences $\{A_n\}$ and $\{c_n\}$ obey well the one-term scaling law

$$\Delta A_n \sim \delta^{-n} \quad \Delta c_n \sim \mu^{-n} \quad \text{for large } n, \tag{57}$$

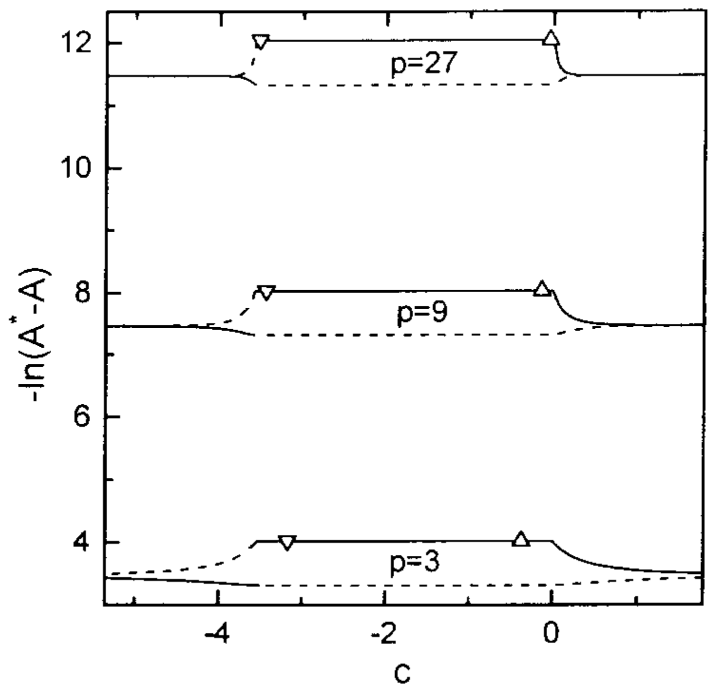

Fig. 2. Stable regions of the in-phase orbits with period $p = 3^n$ ($n = 1, 2, 3$) in two quadratically coupled 1D maps. Each stable region is bounded by solid period-doubling bifurcation curves and dashed tangent bifurcation curves. The up- and down-triangles denote mutually conjugate sequences; the sequence denoted by uptriangles converges to the zero-coupling critical point $(A^*, 0)$, whereas the other one denoted by downtriangles converges to the other critical point $(A^*, -2A^*)$ conjugate to $(A^*, 0)$.

Table 3. In a quadratic-coupling case, we followed a sequence of parameters (A_n, c_n) at which the pair of stability multipliers $(\lambda_{1,n}, \lambda_{2,n})$ of the orbit of level n (period 3^n) is $(-1, -0.5)$. This sequence converges to the zero-coupling critical point $(A^*, 0)$ with the scaling factors shown in the second and third columns.

n	δ_n	μ_n
3	55.264 789 71	2.898 55
4	55.245 771 51	2.965 98
5	55.247 110 93	2.988 62
6	55.247 020 84	2.996 20
7	55.247 026 98	2.998 73
8	55.247 026 56	2.999 58
9	55.247 026 59	2.999 86

where $\delta = 55.247\ldots$ and $\mu = 2.999\ldots$. Note that the value of the coupling parameter scaling factor μ is close to that of the second CE ν_2 ($= 3$), which is in contradistinction to the linearly-coupled case (see Eq. (49)). Hence the first CE ν_1 is not involved in the scaling of the coupling parameter c for the nonlinear coupling case.

Finally, we briefly mention the scaling of the sequence denoted by the down-triangles converging to the conjugate critical point $(A^*, -2A^*)$. It is easy to see that the scaling of the coupling-parameter sequence $\{-c_n - 2A_n\}$ is asymptotically governed by the second CE ν_2 $(= 3)$

$$-c_n - 2A_n \sim C_1 \nu_2^{-n} + C_2 \delta^{-n} \sim \nu_2^{-n} \text{ for large } n, \tag{58}$$

because $\delta > \nu_2$, and C_1 and C_2 are some constants. Hence the scaling behavior of the coupling parameter is the same as that at the zero-coupling critical point. The critical map at the critical point $(A^*, -2A^*)$ is attracted to a fixed map $T^*_{-2A^*}$ conjugate to the zero-coupling fixed map T^*_0

$$T^*_{-2A^*} : x_{t+1} = f^*(y_t), \quad y_{t+1} = f^*(x_t), \tag{59}$$

under iterations of the trifurcation $(M = 3)$ renormalization transformation $\mathcal{N}$ of Eq. (8). The fixed map $T^*_{-2A^*}$ has the same relevant CE's as those of T^*_0. In reality, the straight line connecting the two conjugate points $(A^*, 0)$ and $(A^*, -2A^*)$ is a critical one. The critical behaviors at all interior points of the critical line is governed by another fixed map $T^*_{-A^*}$ (which can be obtained by directly iterating the critical map at $(A^*, -A^*)$),

$$T^*_{-A^*} : x_{t+1} = f^*\left(\sqrt{\frac{x_t^2 + y_t^2}{2}}\right), \quad y_{t+1} = f^*\left(\sqrt{\frac{x_t^2 + y_t^2}{2}}\right). \tag{60}$$

Unlike the two conjugate fixed maps T^*_0 and $T^*_{-2A^*}$, the fixed map $T^*_{-A^*}$ has no relevant CE's. Since $T^*_{-A^*}$ has only one relevant eigenvalus δ like the case of the uncoupled 1D map, the critical behavior at interior points is essentially the same as that for the uncoupled 1D case. The details of the scaling behavior at critical points other than the zero-coupling critical point will be given elsewhere.[23]

4. Summary

The scaling behavior of M-furcations is studied in two symmetrically coupled 1D maps. Using a renormalization method, the dependence of the scaling behavior on M is particularly investigated in the zero-coupling case. It is found that the zero-coupling fixed map of the M-furcation renormalization operator has three relevant eigenvalues δ, α and M. As in the case of 1D maps, the eigenvalue δ governs the scaling behavior of the nonlinearity parameter, irrespectively of coupling. However, the scaling behavior of the coupling parameter depends on the nature of coupling. In a linear-coupling case, it is governed by two CE's α and M, whereas it is governed by only one CE, M, in the case of a nonlinear-coupling case. Taking the trifurcation case as an example, we also study the scaling behavior of the coupling parameter by a direct numerical method and confirm the renormalization results. The relevance of the results on the critical behaviors of M-furcations to the global behavior of coupled maps will be investigated in future.

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